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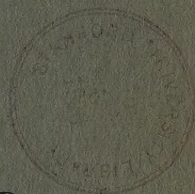
No. 17

THE
PHILOSOPHICAL PRESUPPOSITIONS
OF MATHEMATICAL LOGIC

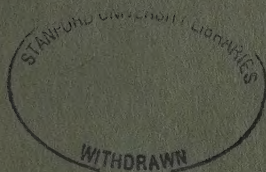
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THE PHILOSOPHICAL PRESUPPOSITIONS
OF MATHEMATICAL LOGIC

A THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF CORNELL UNIVERSITY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY

HAROLD ROBERT SMART, M.A., Ph.D.

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PREFACE.

This Study represents, in the mind of its author, a necessary preparatory step to constructive work in the domain of the logic of science. It was first undertaken at the suggestion of Professor J. E. Creighton, and was presented, about two years ago, under his direction, in partial satisfaction of the requirements for the degree of Doctor of Philosophy. Owing to Professor Creighton's recent illness and death, however, the work of revision, chiefly with a view to simplification, has not had the benefit of his wise counsel and friendly criticism. Professor Frank Thilly has read the revised manuscript, and the writer owes him much in the way of encouragement, criticism, and practical suggestion. Acknowledgment is also due to the other members of the Sage School of Philosophy, and to Professor E. H. Kennard of the Department of Physics in Cornell University, for their personal interest in the work. Finally, readers will observe that this Study is, in a sense, complementary to that of Professor Raymond P. Hawes (No. 15), on *The Logic of Contemporary English Realism*.

H. R. S.

ITHACA, N. Y.

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CHAPTER I.

INTRODUCTION.

The history of the science of logic may be said properly to begin with Aristotle, although many preliminary investigations had been undertaken by his predecessors, and notably by Plato. With the exception of the work of the Stoics, however, little further progress was made in the development of logical theory prior to that revival of interest in the physical universe which is one of the chief characteristics of early modern thought. In the long interim of the Middle Ages the most noteworthy change in the science was that of later Scholasticism, a change which consisted largely in the introduction of an unassimilated mass of verbal subtleties constituting the body of the *Parva Logicalia*.¹ And the chief function of logic, according to Scholastic thinkers, consisted in the service it could render theology, just as in the case of Stoicism and Epicureanism, physics, and logic or 'canonic,' played the part of hand-maids to Mistress Morality.

Under these conditions, hampered both as to its scope and function by certain arbitrary and external limitations, the science of logic, which for Aristotle had existed in its own right and for its own sake, as an unprejudiced examination of the thought forms and processes, under all circumstances in which thinking can or may take place, very naturally declined in character and significance. It became a technique, a narrow specialty, serviceable at best only to the initiated, and, as we have since learned, often employed rather to becloud than to clarify the real issues even on those theological questions with which it was its primary mission to deal.

Now perhaps it was only to be expected that when men's minds turned from the supernatural to the natural, from theology to natural science, there should nevertheless remain traces of the errors of the former period. So it was that change in the object of knowledge altered little, if at all, the fundamental conception of

¹ Adamson, *History of Logic*, pp. 82-3.

a universal method; indeed, the conviction that there was such an unchanging method became, if possible, even more explicit and self-conscious. If, however, we consider more closely the manner in which these two factors—the object of knowledge, and the method by which knowledge was to be obtained—were related to each other in the thought of the early modern period, a steady evolution of the way of conceiving of this relationship may be traced through Descartes, Leibniz, and others, to Kant. Finally, with Kant's doctrine of the categories, thinkers awoke to a realization of the vital interconnection of the form and content of knowledge. In other words, the unity of thought, which heretofore thinkers had conceived of as equivalent to a merely abstract, indifferent unity of method, now came to be looked upon, by Kant and his successors, as a concretely developing unity, expressing itself in, or through, different thought-functions (categories), each of which was naturally fitted to a distinct type of subject-matter. It is noteworthy, moreover, that this conscious recognition of the inseparability of object and method, together with a further analysis of the nature of both of these factors, was mainly the product of Kant's critical examination of the natural sciences themselves, conducted precisely with a view to discovering the philosophical presuppositions necessarily involved in actual scientific achievement. Such an important fact we must not lose sight of in what follows.

Unfortunately, however, we may not here trace this evolution in detail, illuminating as such a study would be. Certain features only will be touched upon, namely, those which necessarily have a place in an intelligible account of the presuppositions and character of mathematical logic. To this end a brief preliminary review of the Cartesian logic will be followed by reference to Hobbes and Locke; and finally, in the following chapter, by a more careful consideration of Leibniz's doctrines. In the light of these preparatory investigations we shall be enabled to study and evaluate, more intelligently than would be otherwise possible, the finished works of the present-day mathematical logicians.

Like Bacon, Descartes recognized the necessity of substituting for the formalities of the Schoolmen an account of the reasoning

process which should furnish a clue to the logical procedure through which man might hope to discover the secrets of nature. It is in this spirit that he sharply criticizes the 'Dialecticians' who give out rules for reasoning mechanically and reaching "a sure conclusion by virtue of the form alone." ". . . the Dialecticians are unable to devise any syllogism which has a true conclusion, unless they have first secured the material out of which to construct it; *i.e.*, unless they have already ascertained the very truth which is deduced in that syllogism."² If such dialectic often entraps those who follow its tortuous windings with regard to truths already known by the light of the 'unaided reason,' it manifestly is unfit to serve as a sufficient or reliable guide to the discovery of new truths.

Unquestionably Descartes is right; the distorted conceptions of Aristotelian logic developed by the later Schoolmen fully deserved the censure here laid upon them. But the very nature of Descartes' criticism of these outworn doctrines clearly indicates the limitations of his own logical principles. For he did not bring in question the notion of a general method of reasoning as such, but only one particular method. Thus, in effect, the very criticism leveled at the old Dialectic of Scholasticism, that it was merely a formal process, a technique, likewise applies in large part, and *mutatis mutandis*, to his own theory. For Descartes, just as for the Schoolmen, the subject-matter of knowledge, and the method employed in attaining knowledge, bear no necessary relation to each other; form or structure and matter or content stand, almost equally in the two cases, in a merely external relationship. If the object or content is no longer the spiritual world but the natural order of the physical universe, so the method or form is no longer that of syllogistic ratiocination but is derived from a narrow interpretation of the nature of mathematical demonstration. As Descartes himself neatly sums it all up in a sentence: "The sciences taken all together are neither more nor less than the human intelligence itself, which is ever one and the same, however different its object-matter."³

² *Reg.* X.

³ *Reg.* I.

In what, then, does this thinker's advance over Scholasticism consist? A reply to this question necessarily involves a brief reference to some of the details associated with his fundamental logical conceptions. Incidentally the answer will serve to bring to light certain of the principles of mathematical logic, though naturally in a form dependent upon the general presuppositions of the Cartesian philosophy. In the first place, as was previously pointed out, all knowledge is, for Descartes, of one structure throughout, "consisting solely in the combination of irreducible elements."⁴ "No knowledge is . . . possible of anything beyond those simple natures and what may be called their combination or intermixture with each other."⁵ That is to say, Descartes denies the possibility of logically evaluating the qualitative aspect of things, and therefore rules this aspect out of genuine knowledge. Or rather, he refers these qualities and varieties of natural objects to geometrical quantity or magnitude.⁶ Thus the only 'qualities' remaining for scientific consideration are extension, figure, and motion.

Secondly, as we also have seen, the structural unity of knowledge entails the identity of method to be employed in its attainment. This method, of course, is the process of mathematical deduction, so-called; a process which starts from intuitively comprehended 'clear and distinct' ideas, or elements, and indubitably attains to absolute truth about complex natural phenomena. Nothing further is needed to apply such a method to any given objects of knowledge except a set of easy rules, which comprise, in short, the logic of scientific procedure. These *Regulæ* (of which only about one-half were ever worked out) lay down specific directions, intended to be easily comprehensible by the ordinary intellect, and designed to convert the reasoning process into a sort of mechanically determined game.⁷ The ideal result of such a game was a chain of statements so clearly interconnected, and so well-ordered with reference to each other, that the reason eventually could pass from one link to the next by a series of 'intuitions' quite as clear

⁴ *Reg.* XII.

⁵ *Ibid.*

⁶ *Reg.* XII, XIV.

⁷ *Reg.* IV.

and distinct as the intuitive apprehension of the original elements forming the starting-point. Such, in fine, were to be the 'certain truths' of science.

This, in briefest outline, was the nature of mathematical demonstration, or proof, as Descartes conceived it. As such it was co-terminous with logical thinking in general. In other words, the art of logic—for it was an art, rather than a science in the proper sense of the word, *i.e.*, a body of organized knowledge—based on the category of quantity, was, for Descartes, the art of thinking according to a method immanent in the very structure of abstract thought itself.⁸—So much, for the present, of this famous conception of a *mathesis universalis*, applicable to all matter capable of scientific treatment; which is also, as we shall come to see, the model of the very latest attempts to devise a set of rules for reasoning on any subject—the ideal goal of contemporary mathematical logicians.

In all fairness to Descartes, however, certain other features of his method, representing his real advance on Scholastic logic, and indicating a more concrete point of view than that of modern Logistic, are worthy of emphasis. If we remember that intuition, or the immediate apprehension of certain fundamental ideas, and deduction, or the mediated apprehension of other (derived) ideas, are held to be, not two distinct processes, but parts of a single method, then it becomes clear that they together constitute a conception approaching to the meaning of the phrase 'scientific insight'. Intuition of clear and distinct ideas, that is to say, was of the very essence of the Cartesian rationalism. For the modern followers of the doctrine of a universal method, on the other hand, the intuition tends to be regarded as an extra-logical source of truth, something quite distinct from the intellect in the proper sense of the word. The Cartesian viewpoint, then, might be said to imply, or at least not to be inconsistent with, though not specifically formulating, the doctrine of a real necessity or rational connection between the contents or data and the method of knowledge. And perhaps this is all that may be demanded of a seventeenth century mathematician, for in the end the comprehension of what

⁸ Cf. Gibson, *Mind*, N. S., Vol. 7, p. 154.

is involved in the logic of science is perforce subject to the character of the existent sciences in question. And as yet, of course, no science worthy of the name, besides mathematics, existed in more than fragmentary outline.

The views of Hobbes, Locke and (later) Condillac are illustrative, in a general way, of the logical conceptions prevalent in the interim that separates Leibniz from Descartes. For these thinkers, just as for Descartes, the ideal type of all knowledge is that which they understood by mathematical. Outside the absolute certainty guaranteed by rigorous mathematical demonstration only probability remained. This however was not to limit all our knowledge to mathematics, since it was still held that the method employed so successfully in that field could also be adapted to other matter. Hobbes, for example, holds that "they also that have not learned the first part of philosophy, namely geometry and physics, may notwithstanding attain the principles of civil philosophy, by the *analytical method*." ⁹ And Locke, although holding that "there can be no science of body," ¹⁰ yet declares that "morality is capable of demonstration, as well as mathematics." ¹¹

Perhaps this is enough to recall the hold which the method of mathematical demonstration, so-called, had on men's minds in the period (the seventeenth century) before physical science had become conscious of its own logical independence. The fundamental conception common to all of the thinkers mentioned, and perhaps inherent in their mutual insistence upon the supremacy of mathematics, together with the implied emphasis on the quantitative aspect of things, is what has come to be called, in our own day, the 'linear' conception of inference.¹² Such differences as that between Hobbes's advocacy of the syllogism and Locke's practical rejection of it, or between the various absolutely fixed starting-points of knowledge, such as 'simple ideas' or 'intui-

⁹ *De Corpore*, Ed. Molesworth, p. 74.

¹⁰ *Essay*, Bk. IV, Ch. XII.

¹¹ *Ibid.*, Bk. III, Ch. XI.

¹² Cf. Bosanquet, *Implication and Linear Inference*. Brunschvicg, in his *Étapes de la philosophie mathématique*, refers to the same notion as that of a 'deduction progressive'.

tions' did not prevent these thinkers from adopting as their own, and uncritically accepting as an adequate account of inference, the Cartesian schema of a demonstrative chain of reasoning, mechanically 'compounded,' or added and subtracted, link by link, from premises or definitions to the conclusion. So much these philosophers gleaned from a logical study of Euclidean and Cartesian geometry. And Locke, perhaps with a vision of modern Logistics in his mind's eye, significantly adds, ". . . what methods algebra or something of that kind may hereafter suggest, to remove other difficulties, it is not easy to foretell."¹³

Let us observe for future reference the essential characteristics of this linear process. Associated with all such conceptions of the nature of inference there are at least three main difficulties, which, it is one of the purposes of this essay to show, are also associated with the procedure of mathematical logic.

First of all there is the selection of a definitive starting-point. Various exponents of the linear-inference view of reasoning have chosen, for this purpose, self-evident truths, simple sensations, simple ideas or impressions; and finally, in recent times, 'primitive' truths or 'indefinables,' and specially 'selected' postulates. In all cases these elementary data are taken as discrete and independent, wholly sufficient unto themselves, and possessed of the absolute certainty which follows from their self-evident character. Yet what rationally compels us to accept as self-evident such *ex hypothesi* merely given desiderata no one has thus far been able to ascertain. Descartes, for example (and after him Berkeley), was forced to depend, in the last analysis, upon God's veracity not to deceive us (a *petitio principii*). And we shall find it difficult not to conclude, from our study of present-day thinkers, that their so-called discoveries or ideal creations are not simply verbal subterfuges to cope with the same difficulty.

Secondly, there is the problem of explaining the movement from one point to another, whether it be along a chain or a series, in the demonstration. On the one hand the whole validity of the process to attain new truth obviously depends upon the original novelty of the various links as they are being forged into a chain. On the

¹³ *Essay*, Bk. IV, Ch. III.

other hand, as we have seen, the truths on which the process finally depends for its certainty are so fundamental that they cannot in their turn depend upon a more ultimate foundation; they must be directly given *a priori*, otherwise we should have a fatal *regressus ad infinitum*. But if, for example, all truth rests in the end upon the will of God, then there can be no necessary or causal connection between things. Indeed, the very word 'thing' loses its meaning, for in principle, clearly, qualities may be changed within a substance and substances may be interchanged with one another. In short, 'anything may come of anything else'. These, at least, were the consequences of this doctrine drawn by Occasionalism; and modern Logistics—we shall see below—seems, naturally enough, to be driven to an analogous conclusion.

Lastly, how are we to understand the general character of this linear process as a development of new knowledge? Does it not suffer from the same defect which Descartes observed of Scholastic logic, namely to presuppose as already known the very discoveries which it enables the operator to make? For the aim of the method is to show how the deductive conclusions, each of which is ultimately to be manifest by itself, may yet somehow reveal themselves as necessary consequences of the first principles. As manifest by themselves, however, these supposedly new truths embody all the characteristics, including their self-sufficiency, of pure and simple intuitions quite apart from any deduction. But if this be so, then the only valuable characteristic of a truly scientific method, namely its power to attain significant results, in the shape of universal and necessary connections, would seem to be conspicuously absent, both from the schematic plans of the earlier methodologists, and from the most ambitious theories of their present-day successors.

If it were possible to include in our study a complete historical account of the doctrines of mathematical logic, considerable space would be reserved for a phase of the subject which may barely receive mention here. This phase consists of the various attempts to construct a so-called logical calculus, *i.e.*, a symbolic method, or technique, analogous to that employed in the mathematical sciences,

of computing conclusions in logic. Beginning with the *Ars Magna* of Raymond Lulle (1235-1315), there has ensued, even down to the present day, a series of works devoted to the structure and operation of such a calculus. The well-known *Principia Mathematica* of Messrs. Whitehead and Russell¹⁴ represents the most modern, and in some respects the most impressive work of this series.

Not to mention lesser names, we have Descartes' own assurance of his interest in the invention of a universal language, which would aid the thinker to guard against mistakes in reasoning on any subject.¹⁵ And of course such an attractive suggestion could not escape Leibniz's attention. Indeed, it at first seemed quite natural to this mathematician-philosopher to believe that it might well be possible to discover by analysis the veritable 'alphabet of human thought,' fixed in the nature of things; just as by an analytical method he had been able to discover the elements of geometrical figures, out of which these figures are developed as functions.¹⁶

But while Leibniz never lost interest in these schemes to develop a 'universal characteristic' or '*filum meditandi*,' to be as useful in theology, jurisprudence, and philosophy, as a similar type of symbols was proving itself to be in mathematics, his attitude towards such mechanical devices changed considerably as he advanced in years and understanding. And in later life, it seems, he never mentions his earlier speculation in this field "without apologizing for it."¹⁷ The essence of the matter seems to be that this thinker finally became aware of the peculiar limitations and difficulties associated with the problem of the adequate symbolic expression of logical operations and terminology. For in the light of his mature philosophical theories, for example the doctrine of the monads, several damaging criticisms of the notion of a logical calculus present themselves. So far from answering the claims of the mathematical logicians¹⁸ that greater brevity, clarity, and rigor of

¹⁴ In 3 vols., Cambridge University Press, 1910-11.

¹⁵ Couturat, *Opusculs et Fragments Inédits de Leibniz*, pp. 27-28.

¹⁶ Gerhardt, *Leibniz: Phil. Schriften*, IV, p. 423; *Monadology*, secs. 34-35.

¹⁷ Lewis, *Survey of Symbolic Logic*, p. 11.

¹⁸ Cf., e.g., *Principia Mathematica*, Preface.

reasoning is obtained by the use of this instrument, Leibniz might well have indicated some of these fundamental weaknesses.

One such weakness is the failure of any symbolism to distinguish properly between a universal term and a particular. Obviously some simple symbol or combination of symbols must represent both; but, if so, the real distinction between them is lost, at least to the reasoner who confines his attention to the mechanical operation of his symbols. Thus it is that Nominalism, and extreme formalism (as in the case of Lulle), often look with favor upon the employment of the calculus; and conversely, one who makes much use of such a tool tends to adopt the exaggerated formalism of a Nominalistic outlook. And Nominalism, of course, is as far as may be from Leibniz's best thought.

Complementary to this fertile source of error is the impossibility of dealing, by means of such a symbolism, with the intension of terms. Leibniz was aware of this difficulty and struggled to overcome it.¹⁹ And it may well be that recognition of his failure in this regard correspondingly lessened his confidence in the efficacy of symbolic modes of expression. Since that time few other attempts have been made to operate a calculus on the basis of the intension of terms, and most writers on the subject frankly recognize the fact that they are forced to adopt the extensional viewpoint. The modern proponents of mathematical logic therefore generally criticize Leibniz on this head, finding in his insistence on the primary importance of intension one of the weaknesses of his logical doctrines.²⁰ Surely this position reveals its own glaring errors to any serious student of logical principles!

Other difficulties might be discussed, for example the notorious inability to represent by a symbolism the processes of induction.²¹ And if space permitted it would be interesting to speculate on how far this infirmity of symbolism is responsible for the appeal to other methods than induction, properly so-called (*e.g.*, Royce's 'selections,' and Mr. Russell's intellectual intuitions), as comple-

¹⁹ Lewis, *Survey*, p. 13.

²⁰ So, *e.g.*, Lewis, *op. cit.*, p. 14; Couturat, *Leibniz*, pp. 23 ff.; Venn, *Symbolic Logic*, pp. xxx and *passim*.

²¹ Cf. Joseph, *An Introductory Logic*, 1st ed., pp. 405, note 1, and 449-450.

mentary to deduction, for the establishment of the universal principles of mathematical logic and pure mathematics. But in our present Study we are concerned rather to attain a clear understanding of the actual principles, the philosophical presuppositions, of mathematical logic, and in this regard, it is plain, the idea and uses of a symbolism can only be ancillary. Accordingly we now turn to a consideration of those principles of Leibniz's philosophy which have an important bearing on our main subject.

CHAPTER II.

LEIBNIZ'S LOGIC.

The significance of Leibniz for the special purposes of our essay it is not difficult to indicate. In the first place, his logical conceptions represent a transition-stage towards a more concrete point of view than that of early rationalism and empiricism. Therefore there is here brought into relief the contrast, embodied in a struggle, between an empty formalism on the one hand, and a developing sense of the importance of content on the other. And secondly, the writings of this philosopher have received a disproportionate share of attention from the authors of the principal modern works on mathematical logic. Hailed by some as the father of the theory in question, by others Leibniz is criticized as by far too much of an Aristotelian in logical matters to count as a real contributor to its formulation. Still other interpreters view his interest in a universal method of reasoning as at most an unimportant incident in his philosophical development. It is plain, therefore, that if we are to study intelligently the philosophical presuppositions of mathematical logic, we must first have made an attempt to evaluate for ourselves, in the light of the best available evidence, these various interpretations and criticisms.

As is very well known, Leibniz, like Descartes, associated his earlier logical studies with the reconstruction of mathematical science. But whereas Descartes' preoccupation with the method of mathematics, or the science of quantity under the form of extension, led him to reject as inadequate the syllogistic procedure of Scholasticism, Leibniz always professed for the logic of Aristotle and the Scholastics the greatest admiration.¹ The question naturally arises, therefore, as to how he came to differ so fundamentally from Descartes in this respect.

It is interesting to note, first of all, that Leibniz, like certain

¹ Couturat complains at the end of his book, *La logique de Leibniz* (pp. 431-441), that Leibniz never fully succeeded in emancipating himself from the logic of Aristotle. Cf. *New Essays*, Bk. IV, Ch. XVII, sec. 4.

modern mathematicians, began his investigation of the first principles of mathematics with arithmetic, whereas Descartes, like certain of the ancient Greeks, devoted his best attention to a study of geometrical principles. Thus in his earliest mathematical work—the *De Arte Combinatoria*—Leibniz assigns to number almost as high a value as it had had in the days of the Pythagoreans. Number is not merely useful in mathematics, it is metaphysically of the greatest importance. “*Cum igitur numerus sit quiddam universalissimum, merito ad Metaphysicam pertinent. . .*”² As the very title of the work indicates, Leibniz holds that all logical relations between concepts may be resolved into quantitative terms. Everything which can be thought about logically stands in the relation of quantitative part to quantitative whole, *i.e.*, in the relation of class inclusion.³ In the same passage he correctly states the factors comprising the logical structure of numerical wholes. First comes the unit, then its repetition resulting in plurality, and finally the result—expressed in the notion of a totality or whole, in which the ‘many’ is grasped as likewise a ‘unity’.

Now even such a whole evidently implies more than a mere repetition or succession of discrete, entirely independent units. In fact, from a bare repetition we can get no whole at all, except by an illegitimate process. That is to say, in order to have a whole—even of quantity—there is required, as complementary to the factor of discreteness, a factor of continuity, implicit in each of the units in the whole because of their common character as its members. Without continuity mere plurality; without discreteness no distinguishable units. This fairly obvious result—we may note in passing—is today, apart from certain of the mathematical logicians, almost universally accepted as a characteristic of quantitative determination.⁴

So far, then, it does not seem that our philosopher has advanced beyond Descartes, unless it be in his more complete analysis of the nature of quantity as such. But he does not long remain satisfied

² Gerhardt, *Leibniz: Math. Schriften*, V, pp. 12 ff.

³ *Op. cit.*, p. 21; Couturat, *Leibniz*, pp. 49–50. This, of course, was in general the standpoint of Descartes in mathematics and of the Nominalists in logic.

⁴ Cf., *e.g.*, Bergson, Bosanquet, Hobson.

with these results. Even in his *De Arte Combinatoria* we find him interesting himself not simply in the development and various applications of the category of quantity in mathematics, but also in a further examination of the relation of mathematics, especially algebra, to logic. And in this respect he soon reaches a most important new conclusion, which involves a complete revision of all his youthful speculations. Namely, it becomes increasingly clear that algebra is not the immediate and complete mathematical expression of logical thinking in general; it is no more, if also no less, than a most important subdivision of it. For whereas algebra is "the general science of quantity," this *Ars Combinatoria* includes also the study of "order, similarity and relations" (a "*doctrina de formulis, seu ordinis, similitudinis, relationis, etc., expressionibus in universum*") ; in short, it is "the general science of quality."⁵

Now by 'quality,' be it noted, Leibniz means, in this connection, "any describable conceptual determinateness in general."⁶ How much more fundamental it is logically than quantity he explains by this comparison: "Quantita seu Magnitudo est, quod in rebus sola compræsentia seu perceptione simultanea cognosci potest. . . . Qualitas autem est quod in rebus cognosci potest, cum singulatim observantur, neque opus est compræsentia. Talia sunt attributa quæ explicantur definitione."⁷ That is to say, the reality of a merely quantitative whole rests, according to Leibniz's new interpretation, upon the reality of its elements, which elements or units receive their logical determination from without, *i.e.*, from the region of sense-experience. And it is to this region of sense-perception, therefore, that the category of quantity properly applies. If, however, this be so, then it would seem that "the derivation of the unit of measurement from 'hypotheses' [in Plato's sense of the word] and rational definitions, which in general are prescribed for mathematical concepts as a condition of their validity, is denied."⁸ But this would be to limit quantity to the domain of the contingent,

⁵ Gerhardt, *Math.*, VII, p. 61; Husserl, *Logische Untersuchungen*, I, p. 221; Cassirer, *Leibniz's System*.

⁶ Cassirer, *ibid.*, p. 167. In this sense it is not to be confused with the sensible qualities revealed in perception.

⁷ Cassirer, quoting Leibniz, *ibid.*, p. 167.

⁸ Cassirer, *ibid.*, p. 167.

and as a result it itself would take on that character. Quality, on the other hand, as expressed in some attribute or essential nature of the extended object is, for Leibniz, a fundamental determination which continues unchanged, however much the magnitude of the object may be increased or diminished.⁹ As such it possesses an aspect of rational necessity and is therefore logically—even metaphysically—significant. It is evident, therefore, that if mathematics, or the science of the quantitative aspect of things, is to retain its character as a rational system of *argumens en forme*, if it is to remain a science at all, that is, some way must be found of relating this apparently sense-limited category of quantity to the rationally grounded category of quality.

Did Leibniz comprehend at once the full significance of this change of viewpoint? Probably not; this could come to him, it would seem, only as a result of further researches into the principles of mathematics, physics and metaphysics. Once fairly started on this road, however, it became impossible, logically speaking, to revert to the old notions, advocated in his early days, of qualityless atoms and discrete, independent particulars.¹⁰ In other words, even this fairly elementary consideration of the logical import of certain mathematical conceptions suffices to reveal the weaknesses of a philosophy of atomism and to awaken the need of somehow transcending, once its logical characteristics become clear, the point of view of quantity.

But perhaps we shall do better to let this mathematician-philosopher indicate the further development of his views in his own way, in close connection with the solution of the scientific problems which occupied so large a place in his thoughts. Take, for a first illustration, the relation between algebra and geometry. "The Geometry of the Greeks was synthetic or synoptic. It dealt with ideal figures as discrete wholes, not taking into consideration the possibility of their being analyzed into elements of which they are combinations or functions. Therefore the relations of figures to one another are considered as external. Each [figure] is what it

⁹ Cassirer, *loc. cit.*

¹⁰ On the development of Leibniz's thought see, *e.g.*, Cassirer, *op. cit.*, ch. on "Die Entstehung des Leibnizischen Systems."

is; no one is regarded as having in it the possibility of passing into another.”¹¹ This, clearly, is the method of sense-perception, pure and simple. Now to Descartes himself belongs the honor of having taken the first step in advance, through his invention of analytical geometry, the basis of which is “the finding of a definite proportion between the space-relations or ratios investigated by geometry and certain numerical ratios.”¹² This discovery, together with Kepler’s re-introduction into mathematics of the notion of infinity, made it possible to establish between the discrete, separate figures of the Greeks a bond of connection, derived from the relative continuity of numerical quantities. Leibniz’s own illustration is that of a parabola which may be considered as an ellipse with one focus at infinity.¹³ The one figure passes into the other by virtue of the “law of continuity.” That is to say, the actual whole, presupposed in all geometrical demonstration, turns out to be, not a particular spatial figure—a parabola or an ellipse,—for such figures merge insensibly into one another, but rather, space as such, as a given object (or presupposition) of mathematical science. It follows that space is not correctly envisaged as composed of a mere sum or aggregate of extended parts external to one another; a little consideration of the notion of infinity suffices to dissipate such a view. Indeed, even the still more abstract numerical infinite transcends the power of a sum of discrete units to express. So that infinity—of space or number—may not be regarded as a whole of discrete parts—of characterless, indifferent units—but as one in which the parts are connected in accordance with a law or principle derived from the intrinsic logical ‘quality’ of number or space as such. Another way of stating the same fact is that the whole, in such a case, is prior to the parts. “. . . there is no infinite number, or line, or any other infinite quantity, if these are understood as true wholes. . . . The true infinite exists, strictly speaking, only in the Absolute, which is anterior to all composition, and is not formed by addition of parts.”¹⁴

¹¹ Latta, *Leibniz; Monadology*, p. 75.

¹² Latta, *ibid.*, p. 77.

¹³ Duncan, *Works of Leibniz*, 2d ed., p. 63.

¹⁴ Russell, *The Philosophy of Leibniz*, quoting Leibniz, p. 244.

Any numerical sum, Leibniz is contending, is a definite number, and hence not infinite but finite. But number, or, more generally, quantity, by itself, without some extra-quantitative qualification, however abstract, from without, possesses no intrinsic determination; there is nothing in the nature of quantity as such to prescribe its determination at one point rather than at another. In other words, it is the very nature, the 'quality,' of quantity, to manifest itself as an indefinite or endless repetition of discrete parts external to one another. Quantity, therefore, must always be determined from without; *i.e.*, any definite quantity must be *quantity of something*; so that there is, properly speaking, no whole of mere quantity, but always of something capable of quantitative characterization.¹⁵ An infinite quantity is therefore, strictly speaking, a contradiction in terms, for the infinite, as such, must be *self-determining*. It is a whole in which the nature of its members is determined by their participation in it, rather than from without. But quantitative parts depend for their being on other parts, *ad infinitum*. On the other hand, the members—for 'members' is a better word than 'part' when we are considering the elements in a 'true whole'—of a true whole reveal the nature of that whole, for they have their being in and through it; and *vice versa*, such a whole reveals itself only in its own members.

What, then, it may be asked, is meant, according to Leibniz, by the terms, infinite space, infinite time, and infinite number? The answer is clear. Space, time and number are not real in the sense in which metaphysical substances are real, but are 'mere ideal things'; their reality, that is, consists in their function of expressing relations between phenomenal objects. ". . . space and time taken together constitute the order of possibilities of a whole universe, so that these orders . . . square not only with what actually exists but also with whatever might be put in its place, as numbers are indifferent to whatever can be *res numerata*."¹⁶ The meaning of this statement, after what has been said with regard to the na-

¹⁵ Incidentally, we may say, though perhaps it would be difficult to find in Leibniz warrant for the assertion, that this characterization usually, if not always, takes the form of measurement.

¹⁶ Latta, quoting Leibniz, pp. 101-102.

ture of quantity, needs little further clarification. On this view, evidently, infinite quantity, in any of its special forms, stands for the expression of a relationship. To say that space is infinite, for example, is a shorthand expression for the fact that spatial relation, externality, cannot be conceived as ever coming to an end—there is always a beyond. To measure space, therefore, would require simply an indefinitely great quantity. Such measurement can never actually be carried out, of course, and hence space is an ‘ideal,’ not a real, thing; by nature a relation, and not a (Leibnizian) substance.

From the point of view of logic only a step now separates Leibniz from the goal of his mathematical studies—the creation of the Infinitesimal Calculus. Number as a sum of finite units, even though they be infinitely many, still presents an aspect of discontinuity. The units, themselves, that is, may be regarded as composed of a similar series of smaller subdivisions. This series finally finds a more adequate mathematical expression in the symbolism of the calculus, and the Law of Continuity at the same time gains a still further significance. Namely, it now comes to mean “the law of the endless relativity of things, the principle of system, of infinite multiplicity in unity [of which] . . . the infinitesimal calculus is an application.”¹⁷

An important consequence of the ideal nature of these fundamental mathematical conceptions—space, time and number—is their relative transcendence of the given-in-sense-perception; while at the same time, of course, it must not be forgotten that they have their origin in, and are inseparable from, such experience. The infinity which may be ascribed to these ‘ideal things,’ therefore, may never be apprehended as a datum of sensory experience, but is rather an unmistakable indication of the merely phenomenal character of that experience. And hence no sensory continuum or physical extension, comprised of *partes extra partes*, could ever replace the ideal factor or logical moment of continuity, which, we have learned, is the inseparable complement of the moment of discreteness implicit in all quantitative determinations. And now, perhaps, we may regard as sufficiently established the meaning and implications of Leibniz’s views on quantity.

¹⁷ Latta, *Leibniz*, p. 83.

We accordingly next proceed to consider certain criticisms, regarding these matters, contained in Mr. Russell's book, *The Philosophy of Leibniz*. The general thesis of that book (in which it is at one with the view of Couturat in *La logique de Leibniz*), namely that Leibniz's theory of monads is "a rigid deduction from a small number of premises,"¹⁸ of a purely formal, abstract nature (therefore satisfying one ideal of mathematical logic), we feel obliged to reject as wholly unsound. Indeed, the conclusion to which our own study points flatly contradicts this view, and maintains rather that the details of the Leibnizian logic and metaphysics, in their mature formulation, rest in general upon his metaphysical, relatively concrete conception of the nature of the monad, as a center of self-activity and self-development. Whether or not Leibniz always remained fully faithful to his best insight in this regard is, of course, another question.

Bound up with Mr. Russell's main thesis—with which we are here only incidentally concerned—is his detailed criticism of Leibniz's theory of judgment as embodied in propositional form. The great weakness in Leibniz's logic, Mr. Russell holds, is due to his persistent insistence on the traditional view that all propositions are reducible in the last analysis to the "subject-predicate form." That is, to judge, according to Leibniz, is the act according to which we qualify a given subject by a predicate or predicates.¹⁹ And the critic comments that consistency with this conception of the subject-predicate form of proposition necessarily leads one to conceive of reality as a whole as either a single substance or an aggregate of substances revealed, in either case, in a series of successive states or accidents.²⁰

Now in holding this view, Mr. Russell maintains, "Leibniz does not differ either from his predecessors or from his successors. Any philosophy which uses either substance or the Absolute [*e.g.*, as well as Leibniz, Kant or Bradley] will be found, on inspection, to depend upon this belief."²¹ It has fallen to the lot of mathe-

¹⁸ *Op. cit.*, Preface.

¹⁹ Russell, *Leibniz*, pp. 12 ff., and *passim*.

²⁰ *Ibid.*, Chs. III and IV.

²¹ *Ibid.*, p. 15.

mathematical logic to free mankind from this 'narrow' doctrine; a feat which it has accomplished by formulating a new "logic of ['external'] relations."²² True, Leibniz does indeed deal with all the main types of relational propositions, such as those of mathematics, but Mr. Russell's complaint is that he endeavors to reduce them also to the subject-predicate form. "Thus Leibniz is forced, in order to maintain the subject-predicate doctrine, to the Kantian theory that relations, though veritable, are the work of the mind."²³ And the relevancy of considering Mr. Russell's criticisms here is due to the fact that the most important examples of such relations are space, time and number—the domain of mathematical science. Let us study these criticisms in order.

1. With regard to the first point, namely that substance, thought of as subject, represents a simple sum of attributes, assigned to the predicate, Leibniz himself has this to say: "In distinguishing two things in substance, the attributes or predicates and the common subject of these predicates, it is no wonder that we can conceive nothing particular in this subject. It must be so, since we have already separated from it all the attributes in which we could conceive any detail. Thus to demand something more in this pure *subject in general* than what is necessary to conceive that it is the same thing . . . is to demand the impossible and to act contrary to our own suppositions . . . [in] conceiving separately the subject and its qualities or accidents. We could apply the same difficulty to the notion of being and to all that is. . . ."²⁴ So much for Leibniz's views about the thing-like conception of the subject of a proposition. There could be no more definite rejection of the notion that the proposition is primarily an embodiment of the category of substance and accident. Incidentally, it should be noted that these remarks constitute an implicit criticism of the law of abstract identity as a source of new truth.

Furthermore, Leibniz develops his own positive theory of the nature of the subject term. He explains that this conception of a

²² *Principles of Mathematics*, esp. pp. 95–100. Cf. also other writers on the subject: Royce, Professor Lewis, Professor Spaulding.

²³ *Op. cit.*, p. 14.

²⁴ *New Essays*, Bk. II, Ch. XXIII, sec. 2.

subject, ". . . entirely slender as it appears," when viewed as a material substratum for attributes to adhere to, is not, when rightly viewed, "so empty and sterile as you think. It gives rise to many consequences of greatest importance in philosophy, and which are capable of giving it a new aspect."²⁵ Thus the 'empty and sterile' idea, which Leibniz attributed to Locke, and which Mr. Russell attributes to Leibniz, the last-mentioned thinker specifically rejects, in favor of the notion of a unitary, spontaneously active monad—quite a different type of 'substance'. It is not the subject of a proposition in the sense of a particular thing possessing only sense-qualities or adjectival attributes; nor merely a pure numerical simplicity, externally exclusive in its self-consistency; it is, essentially, a "center or point, in which, quite simple though it is, there exists an infinite number of angles formed by the lines which meet in it."²⁶ "Thus the combination of self-consistency with consistency in relation to the whole is what Leibniz means by the character of the monad as at once exclusively individual and representative or perceptive of the whole universe from its particular point of view."²⁷ Abandon Leibniz's pluralistic hypothesis and there emerges the germ of the notion of a self-developing totality, from which, in the fashion already intimated, certain subjects and predicates come to be distinguished relatively to one another within the whole. For Bradley, for example, judgment was never a predication of certain qualities *of* Reality, but rather a process of distinguishing attributes which are all taken as falling *within* Reality, or the concrete whole of things. Such an attribute, it is evident, is quite a different thing from a mere adjective. While the distinction is not difficult to understand, it is nevertheless fundamental, and unfortunately the critic has failed lamentably to comprehend its significance.

2. As for the so-called relational propositions (*i.e.*, propositions expressing mathematical relations), which Leibniz admittedly recognizes but attempts also, according to Mr. Russell, to reduce to the 'subject-predicate form,' there seems to be a good deal of

²⁵ *Loc. cit.*

²⁶ *Principles of Nature and Grace*, sec. 2.

²⁷ Latta, *Leibniz*, pp. 70-71.

misunderstanding, in this latter thinker's mind, as regards both their interpretation and their assumed 'mental' origin.

Perhaps there is no better way to approach a study of these problems than the one which Leibniz himself supplies. He says: "The ratio or proportion between two lines L and M may be conceived three several ways; as a ratio of the greater L to the lesser M ; as a ratio of the lesser M to the greater L ; and lastly, as the ratio between L and M , without considering which is the antecedent, or which the consequent. . . . In the first way of considering them, L the greater, in the second, M the lesser, is the subject. But which of them will be the subject in the third way . . . ? It cannot be said that both . . . L and M together are the subject of such an accident; for, if so, we should have an accident in two subjects, with one leg in one and one in the other; which is contrary to the notion of accidents. Therefore we must say that this relation . . . is indeed *out* of the subjects; but being neither a substance nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful."²⁸ In short, according to Leibniz, the notion of substance and accident does not apply in such a case. And as for the notion of relations, which does apply, the reality which can be assigned to such ideal things by science is not an ontological, substantial one, but one which consists in the 'truth of relations,' as expressed, for example, in a scientific law or principle. And this is a fundamental contention, not only of Leibniz, but of all those referred to by the mathematical logicians as exponents of the doctrines of traditional logic. Relations and terms, they maintain, can only be distinguished relatively to each other in all kinds of wholes, implicitly or explicitly assumed, under all sorts of prescribed or definitely determined conditions. And neither the one nor the other can be viewed as existential some-whats, whose character is wholly independent of modification in the other.

Now Mr. Russell has subjected the passage just quoted to considerable criticism, both in his *Philosophy of Leibniz* and again in his *Principles of Mathematics*.²⁹ He points out that it is "of

²⁸ Duncan, *Works of Leibniz*, pp. 359-360.

²⁹ In the former work, pp. 12-16, and in the latter, pp. 222-224.

capital importance for a comprehension of Leibniz's philosophy. After he has seemed, for a moment, to realize that *relation* is something distinct from and independent of subject and accident, he thrusts aside the awkward discovery, by condemning the third of the above meanings as a 'mere ideal thing'." ³⁰ Mr. Russell himself goes on to argue for the "independent reality" of relations; insists indeed (what in a sense no one is concerned to deny) that "judgments of subject and predicate are themselves relational." ³¹

But is it quite true, we have to ask, that Leibniz 'condemns' relations to a limbo of unreality in just the sense implied by his critic? And does this critic himself really solve any problems by his own ontological interpretation of the nature of a relation? To the second question we shall have to return again, since a full answer presupposes acquaintance with the complete doctrine of relations as set forth by the mathematical logicians. But a partial answer Leibniz himself supplies, and this will serve as sufficient excuse for considering it here.

In his *Principles of Mathematics*, the author seems to forget Leibniz's argument advocating the third alternative, and proceeds to criticize, as the 'monadistic' theory, the first two ways of considering the relation of *L* and *M*. Since, however, Leibniz specifically rejects all but the third way—a result which would naturally follow from his doctrine that in ideal constructions the part presupposes the whole—it is difficult to find any relevancy in the critic's conclusion, namely that (the assumed) exponents of the theory "are forced to admit what the theory was designed to avoid, a so-called 'external' relation, i.e., one implying no complexity in either of the related terms." ³²

Mr. Russell next proceeds to deal with the third way, really advocated by Leibniz, but which his critic labels a 'monistic' theory. This theory, we learn, "holds that every relational proposition [symbolized by] aRb is to be resolved into a proposition concerning the whole which 'a' and 'b' compose—a proposition which we may denote by $(ab)R$." In this way of regarding the ratio "we

³⁰ *Leibniz*, p. 13.

³¹ *Ibid.*, p. 15.

³² *Ibid.*, p. 224.

do not consider which is the antecedent and which the consequent; and it is indeed sufficiently evident that in the whole (ab) as such there is neither antecedent nor consequent. In order to distinguish a whole (ab) from a whole (ba) , as we must do if we are to explain asymmetry,³³ we shall be forced back from the whole to the parts and their relations. For (ab) and (ba) consist of precisely the same parts, and differ in no respect whatever save the sense of the relation between 'a' and 'b'. 'a is greater than b,' and 'b is greater than a,' are propositions containing precisely the same constituents, and giving rise therefore to precisely the same whole; their difference lies solely in the fact that *greater* is, in the first case, a relation of 'a' to 'b,' and in the second, a relation of 'b' to 'a'. Thus the distinction between an asymmetrical relation and its converse is one which the monistic theory of relations is wholly unable to explain."³⁴

Here we have a typical criticism of traditional logic as set forth by an able exponent of 'symbolic' or mathematical logic. Incidentally, it is unconsciously directed against one who, it has often been claimed, was one of the progenitors in this field, and seems to hint that at heart Leibniz was not so much in sympathy with modern 'Logisticians' as some of them would have us believe. It is a simple matter to point out the fallacy in Mr. Russell's statement, a fallacy evident, indeed, even in the abstract symbolism which he employs. Consider for example the phrase "the whole which 'a' and 'b' compose—a proposition which we may denote by $(ab)r$." How does it happen that at one and the same time we have a whole composed of 'a' and 'b' simply, while its expression, even in symbolic terms, requires an additional symbol, namely 'r'? Clearly, it would seem, the 'whole' composed of 'a' and 'b' in simple spatial juxtaposition is not equivalent to the whole here in question. In other words, this whole *is not* composed of the sum of the parts 'a' and 'b' simply, but of these (distinguishable) parts as (inseparably) related in a certain definite manner. Or, as Leibniz explained, the real subject of the proposition 'a is greater than b' is neither 'a' nor 'b,' nor yet both together, but rather (*e.g.*)

³³ Cf. below, Ch. IV.

³⁴ *Op. cit.*, p. 224. For Mr. Russell, the 'converse' of 'greater' is 'less'.

space as such, as comprising, in an ideal whole, all such systems of terms and relations in virtue of, and as constituting, its specific logical 'quality'. And the different distinctions of sense, which Mr. Russell requires, is furnished precisely by holding these factors together in a concrete unity such as finds expression in the proposition ' a is greater than b '; which also means, in virtue of the very nature of spatial relations, as revealed by scientific experience, that ' b is less than a '. In other words, it is a fundamental thesis of the so-called monistic theory, held by Leibniz and his successors, to maintain that relations and terms, while distinguishable, are inseparable, so that it is wholly false to say that the two statements here involved give rise to "precisely the same whole." Even Mr. Russell would not say that ' a is greater than b ' is equivalent to ' b is greater than a '. He would not say so, that is, if ' a ' and ' b ' be allowed to possess any significance at all.

The two grounds, then, which impelled Mr. Russell to attempt to re-analyze the proposition, namely the inadequacy of the old conceptions to express mathematical relationships, and the general objection to the substance-accident type of reasoning, have—so far as Leibniz is concerned, at least—no justification in fact. For he has neither proved that mathematical inference in so far demands another interpretation of the proposition than that furnished by traditional logic, nor that Leibniz or his successors actually held the views attributed to them by their critic with regard to the nature and function of the proposition. Neither does he succeed better, we shall be forced to conclude, when he attempts to construct a positive theory of his own.

3. The objection, finally, against the attribution of a merely ideal, mental character to these fundamental mathematical relations may easily be answered, if not out of Leibniz's own mouth, then in the plain implications of his views as developed by Kant. In a sense, that is, both relations and the terms which they relate may be said to be the work of the mind, and yet we must also maintain—which is what the critic in his own way stands for—that they as surely belong to the objective order and are no mere subjective, mental phantasma. To hold both of these positions at once, however, involves a conception of mind, as the expression of

objective reality in ideal form, which—we shall have occasion to note again below³⁵—Mr. Russell seems never to have contemplated and at all events has never criticized.

We have now carried our study of Leibniz far enough to formulate certain conclusions. We have seen, namely, that he was at least partially successful in his attempt to avoid two errors. Alike in his studies on the foundations of the exact sciences, and in his philosophical speculations, he has advanced beyond the narrow rationalism of Descartes on the one hand, and the mere empiricism of Locke on the other. Thought—even scientific thought—does not spin its own constructions out of its own abstract inwardness, as Descartes had supposed, but comes to discover new and higher notions only through its operation on the material presented, in the first instance, by sense-experience.³⁶ On the other side, sense-experience, as such, possesses no rational or metaphysical significance; only after it has been transformed into the material of science does it gain such reality as it is capable of possessing. So, in the end, only 'contingent' truth can be derived from mere sense-experience, and hence empiricism, if strictly adhered to, will carry us only so far on the road to knowledge. "Since the senses never give anything but instances, that is to say, particular truths, . . . they do not make known to us what these sense qualities are;"³⁷ they do not, *i.e.*, reveal any bond of connection, of systematic relation to one another. "Mere empirics" [including animals] put their trust in the testimony of the senses and form "concatenations" of ideas which are "only a shadow of reasoning . . ."; only "connections of imagination and passings from one image to another, because in new circumstances which seem to resemble others which have occurred before we expect anew what at other times we found along with them, as if things were actually connected together because their images are connected in memory."³⁸ Linear inference (which, we saw above, was in gen-

³⁵ Cf. Chap. V.

³⁶ Cf. *New Essays*, Bk. I, Ch. I, sec. 1: "... one can say in a certain sense that the external senses are in part cause of our thoughts. . . ."

³⁷ Latta, quoting Leibniz, p. 362.

³⁸ *New Essays*, Intro.

eral the common ideal of Bacon, Descartes, Hobbes and Locke) practised upon a series of particulars, taken as given and discrete, gives no "necessary and infallible truth." "Demonstrative science" alone can find "sure connections having the force of necessary consequences" so that we can determine events in advance [the predictions of science] "without having to experience the sense-connection of images."³⁹

Here we have in effect a criticism of those later doctrines of associationism, which perhaps would never have arisen had this passage been taken to heart by their advocates. And although Leibniz still apparently persists in the views of his age that all true science must be demonstrative and deductive in character, after the model of mathematics, yet if the sentences just quoted may be taken as typical, it is evident that their author has gained a new insight as to the ground of certainty of mathematical truth. A series of intuitions is no longer viewed as an adequate test of such truth, and neither is the demonstration envisaged simply as a mechanical movement according to rules, from point to point along a chain. Instead of these ideals, Leibniz seeks for 'sure connections,' according to rational principles (*e.g.*, the law of contradiction), so that ground and consequent mutually condition each other in what may properly be called a logical system. One step is related to another by rational necessity, and thus the whole demonstration reveals itself as an interconnected body of propositions. In modern phrase, the principle here at work is that of systematic coherence, rather than that of linear inference.⁴⁰

Now demonstration, in its function as a technique, can be carried on in the fields of physics and jurisprudence, as well as in mathematics, without a clear knowledge of the respective first principles or presuppositions of the sciences in question.⁴¹ Never-

³⁹ *Loc. cit.*; cf. also *Monadol.*, secs. 26-29.

⁴⁰ Leibniz also seems to have recognized the need of an "art of invention," *i.e.*, of induction; but we must confess that he did not go far towards making clear and definite the procedure of the 'art' in question. Cf.: "Das durch die Logik nichts erfunden, kann ich nicht zugeben; alles was durch den Verstand erfunden, ist durch die guten Regeln der Logik erfunden, obschon solche Regeln anfangs nicht gleich ausdrucklich aufgezeichnet oder zusammen geschrieben gewesen." Erdmann, *Leibniz Opera*, p. 424.

⁴¹ *Discourse on Metaphysics* (Montgomery transl.), p. 16. Cf. also letter to Arnauld, Nov. 28-Dec. 8, 1686.

theless, there must be a science which shall furnish a logical account of the derivation of these principles or "definitions," as Leibniz calls them, from which the deductive demonstration proceeds to its necessary conclusions. The special function, then, of logic, or general science, is to prescribe the universal characteristics of such definitions.⁴² So Leibniz is led to explain that "real logic," which differs little from "the true metaphysics," is not merely an instrument or technique, but that it has to do, rather, with the universal principles of all branches of reasoning, and lays down the general rules for distinguishing the true from the false.⁴³

In the light of the previous exposition, however, we should expect that a definition would depend upon an analysis of the object to be defined, with a view to determining its ultimate nature in 'qualitative' terms. As Leibniz says, "in order to know a thing we must consider all the requisites of that thing; i.e., all that which suffices to distinguish it from every other thing. This is what is called definition, nature, reciprocal property."⁴⁴ This is true of all objects of knowledge whatever, whether they be considered in mathematics, physics, ethics or metaphysics.⁴⁵

Moreover, a (real) definition not only reveals the essential nature or quality of any object; in virtue of this very fact it also expresses the manner of its genetic derivation or construction.⁴⁶ As Dr. Cassirer interprets Leibniz's view, "The real definition is therefore always a 'causal' definition, in the sense that thought shows itself to be the cause of its object. This is made clear by the example of the definition of a circle; the possibility of a circle remains doubtful so long as it is conceived as characterized by merely sensible properties; its reality is established by means of the constructive rule [depending upon the nature of space] according to which it may be produced."⁴⁷

⁴² Couturat, p. 184.

⁴³ Gerhardt, *Phil. Schrift.*, IV, p. 137; cf., also, *New Essays*, translator's note, pp. 422-3; and Cassirer, *passim*.

⁴⁴ Duncan, *op. cit.*, p. 296.

⁴⁵ Gerhardt, *Phil. Schriften*, V, *passim*.

⁴⁶ Cassirer, *op. cit.*, p. 112. Evidently Leibniz is here following Aristotle.

⁴⁷ Cassirer, *loc. cit.* Cf. Leibniz, *Metaphysics*, Eng. transl., p. 42.

Leibniz significantly contrasts his doctrine with that of Hobbes. Hobbes held that definitions represent meanings arbitrarily assigned to words, to which we must hold fast in all subsequent reasonings. And truth consists in binding these words together into propositions and syllogisms according to the law of contradiction. "A syllogism is nothing but a collection of the sum of two propositions joined together by a common term. And as a proposition is the addition of two names, a syllogism is the adding together of three."⁴⁸ Nominal definitions, then, *ex hypothesi* contain only 'marks' or sensible qualities.⁴⁹ Leibniz will have none of this. "Definitions are not arbitrary as Hobbes believed, and we cannot form ideas as we like, though it seems the Cartesians are of this opinion. For it is necessary that these ideas which we undertake to form should be veritable, *i.e.*, possible, and that the ingredients which we put into them should be compatible *inter se*."⁵⁰ Put in slightly different language, but with just as emphatic conviction, is the assertion, in the *New Essays*, that "when one demands *præ-cognita* in the sciences, . . . which serve to ground science, he demands *known principles* and not arbitrary propositions, . . ." as even Aristotle recognized.⁵¹

What Leibniz evidently intends to teach here is that a quantitative aggregate or combination of ideas forms no real synthetic concept expressing the nature of an object. Such an aggregate does not show whether the object be 'possible' or not; that is, whether the complex idea of the object contain as parts simple ideas which contradict each other or not. As an example of an impossible or contradictory concept he mentions a regular decahedron which would be defined nominally as a body comprised of ten planes. In reality there is no such figure, as the real definition, or mode of its construction, would show.⁵²

⁴⁸ *De Corpore*, Ed. Molesworth, p. 48.

⁴⁹ Cf. above, Ch. I. We must not confuse sensible qualities with the essential quality or essence; the former are the product of sense-experience; the latter is the result of rational analysis of the concept.

⁵⁰ Russell, quoting Leibniz, p. 208.

⁵¹ Bk. IV, Ch. XII, sec. 4.

⁵² *New Essays*, pp. 315-316. Cf. "Plusieurs qui ont philosophé en mathématiques sur le point et sur l'unité se sont embrouillées faute de dis-

So much for the logical nature of a (constructive) definition and its corollaries. From a consideration of this subject we are naturally led to the process of demonstration itself, which is based upon definitions. This process is guided by two supreme principles, which control its application to any object of knowledge, and form its necessary and sufficient presuppositions. "In demonstration I use two principles, of which one is that what implies contradiction is false, the other is that a reason can be given for every truth (which is not identical or immediate)." ^{52a}

In this connection Mr. Latta holds that Leibniz had a far deeper philosophical insight than Descartes (and Spinoza) while apparently teaching a similar doctrine, while Dr. Cassirer emphasizes Leibniz's advance over Hobbes.⁵³ "The logic underlying the philosophies of Descartes and Spinoza was a logic of abstract self-consistency. In their view all real knowledge must be of one kind. . . . This was necessarily involved in the position that there is no appeal beyond the witness of consciousness to itself. 'The order and connection of ideas is the same as the order and connection of things.'" ⁵⁴ The principle by which all ideas are linked together is that of self-evidence or non-contradiction, "which is simply another way of describing the witness of consciousness to itself. . . . In other words, they held that self-consciousness is self-consistent, that *it never contradicts* itself." ⁵⁵ This, when translated into logical language, is a bare tautology, that 'A is A,' or that 'A is not non-A'. Evidently the principle of contradiction conceived of in this purely psychological way does not take us far. And neither does Hobbes's absence of contradiction among conventional signs and nominal definitions succeed better in helping us attain the desired goal of objective truth.

tinguer entre la Résolution en notions et la Division en parties. Les parties ne sont pas toujours plus simples que le tout, quoy'quelles soyent toujours moindres que le tout." Gerhardt, *Phil. Schrift.*, III, p. 583. That is to say, logical unity is not resolvable; there are no separate parts the idea of which is simpler than it. Unity is divisible but not resolvable.

^{52a} Gerhardt, *Phil. Schrift.*, VII, p. 199.

⁵³ Latta, *Leibniz*, pp. 58-63; Cassirer, *Leibniz*, pp. 109-116.

⁵⁴ Latta, *ibid.*, p. 58; above, Introduction.

⁵⁵ Latta, *ibid.*, p. 59; above, Ch. I.

In almost identical phrase Mr. Latta and Dr. Cassirer explain what, on Leibniz's view, was lacking in these doctrines. "All real self-consistency must be mediate; it must have grounds. It must spring from the specific nature of the self-consistent thing. . . . Self-consciousness, then, is really self-consistent only in virtue of its being a definite system, a self-revealing process or development, which contains within itself the ground or reason of its self-consistency, and the ground or reason of existence."⁵⁶ These remarks need little further substantiation; they are too clearly implied by Leibniz's metaphysical principles. We may, however, seek independent support from another interpreter. Dr. Cassirer also maintains that formal logic alone cannot show us how to apply the law of contradiction to any given material; for this we must return to the fundamental principles of scientific experience, which is specially adapted to deal with such material. "The examples which Leibniz introduces plainly illustrate this contention. Namely, they refer always to the concepts of mathematics and mechanics, and the standard according to which the objectivity of these concepts is measured, is sought for, therefore, wholly in the positive and particular principles of these sciences. Thus the greatest number is an 'impossible' concept, not in a formal-analytical sense, but because it—as Leibniz explains in detail—contradicts a fundamental axiom of number, that is, the quantitative aspect of things. Similarly, of the geometrical concept of a greatest circle, and of the mechanical concept of a greatest velocity."⁵⁷ In short, as both of these writers insist, self-consistency in the domain of science is, for Leibniz, a concretely systematic, rather than an abstractly formal principle. And after all, this is quite in keeping with the Leibnizian notion of the nature of a definition, as already outlined. How foreign it is to the spirit and teachings of mathematical logic we shall see as we proceed, while it is at once evident that Leibniz's interpretation represents a real advance over that of Descartes and Hobbes. Moreover, it must not be forgotten that the difference is not one of mere formal logic. For outwardly, the formal enunciation of the principles is much the

⁵⁶ Latta, *op. cit.*, pp. 59–60.

⁵⁷ *Op. cit.*, p. 112.

same; it is the firmer metaphysical foundation, evidently, which is responsible for the contrast.⁵⁸

The second logical principle, which, in Leibniz's opinion, supplements that of non-contradiction, he calls the principle of sufficient reason. It is defined as that "in virtue of which we hold that no fact can be found real or existing, no statement true, unless there be a sufficient reason why it should be so and not otherwise, although these reasons can not very often be known by us."⁵⁹ That is, in the case of necessary truths, for which the principle of contradiction holds, the reason can always be given, but in the case of contingent truths, for which the principle of sufficient reason holds, we can only be sure that there *is* a sufficient reason, but cannot always know precisely what that reason is. Stated in terms of definition, there are cases where the analysis into simple elements is only potentially complete, or complete only in the mind of God. The first principle, that of contradiction, has as its metaphysical basis the existence of real substances, even more indivisible and more ultimate than physical atoms. The complementary principle, that of sufficient reason, metaphysically interpreted, yields, not a mere static aggregate of possibly contradictory parts, but a real, developing whole, a perfect system of consistent or 'compossible' substances, each one of which in some way ideally mirrors the whole. The most obvious defect in this account is inherent in the view that mere analysis is ever capable of yielding any such ultimate simples. For analysis is not the name of a complete process, as Leibniz was inclined to suppose, but a factor which finds completion in the idea of synthesis. And Leibniz may well have been led to his position by the equally one-sided advocacy by the Cartesians, of the other factor, namely synthesis, associated with their interpretation of inference as deductive. By abandoning either of these false abstractions—which involves also the relinquishment of the notion of absolutely simple elements—the way is cleared for a reunion of Leibniz's two principles, and a similar revision of his *Monadology*. Thus it is recognized today that the principle of sufficient reason applies to all scientific inference, and that the prin-

⁵⁸ Cf. Windéband, *Gesch. d. Phil.*, I, pp. 474 ff.

⁵⁹ *Monadology*, secs. 36, 37.

ciple of contradiction in its abstract form represents no concrete criterion of truth.⁶⁰

What now is the attitude of the mathematical logicians towards Leibniz's employment of these two principles? A representative criticism from the point of view of mathematical logic will be found in Couturat's previously mentioned work, *La logique de Leibniz*.

Couturat interprets the principle of sufficient reason as a mere reciprocal to the principle of identity or contradiction. "The principle of identity says: every identical (analytic) proposition is true. The principle of sufficient reason affirms . . . that every true proposition is identical (analytic)." The result is that all truths come under the principle of (abstract) identity.⁶¹ From this point of view also, definitions function as media between the principle of identity and the axioms of the special sciences.

Clearly, Leibniz is here pictured as a consistent representative of what we shall discover are the characteristic doctrines of mathematical logic. The reasoning process is carried on according to a linear schema from the most general and abstract principle at the one end to the final particular deductive conclusion at the other. Little wonder that Couturat himself should describe the results of such a process as this, which he ascribes to Leibniz, as 'barren'!⁶²

Against this interpretation, however, several objections present themselves. Not to mention Leibniz's specific rejection of such 'concatenations,' if our account of the function of definitions is correct, it would seem that not even in the mathematical sciences can the principle of contradiction or identity serve alone as a sufficient basis for deduction. Or, as Dr. Cassirer expresses it, Leibniz's real doctrine seems in general just the reverse. That is, "the identical propositions comprise the technical means [associated with demonstration] for the unfolding and analysis of the concrete content, which is independently created by the original 'causal' definitions of reflective thought."⁶³ And while it is true

⁶⁰ Cf., e.g., Bosanquet, *Logic*.

⁶¹ "Sur la métaphysique de Leibniz," *Rev. de Méta.*, X, pp. 1-25. *Op. cit.*, p. 187.

⁶² *Op. cit.*, pp. 439-441.

⁶³ *Leibniz*, p. 544.

that the analysis of contingent truths leads on from link to link in the chain of existents, nevertheless, the principle of contradiction alone can never account for, or verify, existence in general.⁶⁴

This distinction between necessary and contingent truths—expressed by Leibniz in logical terms as a distinction between the principle of contradiction and that of sufficient reason—is grounded not in a psychological or subjective and therefore empirical limitation, but is in principle a necessary condition or presupposition of scientific knowledge and its objects. “The difference between Geometry and Physics, as Leibniz writes to Bernouilli, ‘does not rest upon the momentary state of our knowledge, but is based upon things themselves’.”⁶⁵ And to Arnauld Leibniz explains that “in the ultimate analysis of the principles of physics and mechanics, . . . it is found that these assumed principles cannot be explained solely by the modifications of extension, and the very nature of force calls for something else.”⁶⁶

What is this but to say, if in an inadequate and abstract way, that each science must define its own objects, prescribe its own subject-matter, independently, and that it cannot take them from some external source, such as formal logic or a general mathematics? Ultimately, of course, these scientific principles themselves point to certain metaphysical presuppositions which thus prescribe the limitations of science. In sum, it seems very evident that Couturat has consistently confused two things—a confusion, we may hint in anticipation, which underlies the whole point of view of mathematical logic in its relation to mathematics. Namely, he interprets as the essential thesis of the Leibnizian philosophy what at most represents only its “general framework, which equally comprehended all the great systems—Descartes, Spinoza and Hobbes—of the seventeenth century.”⁶⁷ There is, indeed, a certain resemblance between all these systems, but a resemblance due, not as Couturat supposed, to absolute identity of plots, but rather to similarity of stage-setting for their representation. This,

⁶⁴ Latta, pp. 67 ff., and *passim*; also Russell, *Leibniz*, pp. 25–39.

⁶⁵ Cassirer, *op. cit.*, p. 545.

⁶⁶ Letter to Arnauld, Nov. 28, 1868.

⁶⁷ Cassirer, *op. cit.*, p. 547.

however, is but to say that Leibniz inherited from the age in which he lived the limitations and inadequacies of his thought, while his own thinking represents a partially successful effort to transcend these boundaries.

However that may be, it is admitted on all sides that Leibniz failed, for one reason or another, to realize the goal, as conceived both by Descartes and by present-day authorities on mathematical logic, of a *mathesis universalis*. For some interpreters of his system this failure is to be ascribed to his growing realization of the false nature of that ideal. By other commentators it is maintained that he lacked the proper insight into details to accomplish more than a vague formulation of the general principles of the science in question. And our own conclusion would be that there may be traced a steady development of his thought, away from abstractness and towards concreteness, and a consequent abandonment of his earlier visions. That he never completely escaped from the toils of such abstraction, however, his faith in the power of analysis to arrive at certain truth, at least in some instances, is sufficient evidence. But on the other hand, his clear recognition of the need of other principles and categories than those formerly held to be sufficient in order to systematize the world of *phenomena bene fundata*, represents an undeniable advance, only the more significant because of his actual contribution to scientific thought. The definitive rejection of the linear-inference conception of reasoning and its replacement (at least in principle) by the idea of self-development, sustained by a rational process of systematic inference, then appears as but the natural extension of the more concrete philosophical conceptions of Leibniz's mature system.

CHAPTER III.

THE MODERN ORIGIN AND DEVELOPMENT OF MATHEMATICAL LOGIC.

There is much truth in the contention that in philosophy, at any rate, old ideals and problems never perish absolutely; that from age to age they appear and reappear in endless succession, while what at first glance seems new in them reveals itself on examination as only a reformulation, a re-statement, of certain apparently inevitable attitudes of the human spirit towards Reality.

Two such problems are those concerning the nature of infinity, and the relation of the discrete to the continuous. Speculative philosophy, the philosophy in general of Plato, of Leibniz, Kant and Hegel, has always prided itself on its ability to state these problems clearly, if not to solve them definitively. But Realism, the 'common-sense' philosophy, has always had special difficulties to contend against with regard to such matters. In his admirable work, *Les étapes de la philosophie mathématique*,¹ M. Brunschvicg presents some of these difficulties, in their modern form, as follows: "How are we to admit the transcendent existence [in the sense demanded by Realism] of mathematical objects, when mathematical science rests upon the fundamental ideas of infinity and continuity?" The philosophy of common sense, which takes for real things as it finds them, presupposes that the entities so intuited should be of a finite number and distinct the one from the other—that they should constitute a discrete plurality.

Now Leibniz and Kant, though thoroughly imbued with the spirit of Aristotelian realism, were nevertheless forced by the successes of contemporary mathematics "to break down the limitations of the ancient ontology, to treat continuity and infinity as 'ideal' things, and to make of space and time a system of relations or forms of the mind."² So much for the problem, from the point of view of modern idealism, and its already sufficiently indi-

¹ *Op. cit.*, pp. 390-391.

² *Ibid.*, p. 391.

cated solution. For the realists of the latter half of the nineteenth century, on the contrary, there was apparently nothing left but to limit actuality to the domain of the finite and the discontinuous. No wonder, then, that a realistically-minded thinker of our own day should take particular interest in a new Theory of Aggregates, which apparently gives to these conceptions of infinity and continuity a new mathematical interpretation. It must be remembered, however, that we found, in the preceding chapter, that so far as interpretative criticism of Leibniz was concerned, contemporary exponents of mathematical logic were largely unsuccessful in their attempts to understand and evaluate the significance of that thinker's contribution to thought on these and related matters. In this chapter, therefore, we shall have to undertake a complementary task with regard to their speculative inferences from the above-mentioned developments in modern mathematics.

A necessary introduction to this critical investigation, evidently, is to set forth as briefly and simply as possible what are the foundations of this new mathematical construction. Then we shall be in a position to examine the positive claims of the 'logisticians' with regard to the philosophical significance of this assuredly important scientific advance. More particularly, the question must be asked, in the light of such developments, to what extent must the philosopher modify the doctrines, inherited from Leibniz and his successors, concerning the properly philosophical meaning of the conceptions of infinity, continuity and quantity. And finally, we shall have to consider (in succeeding chapters) what may be the value, for mathematics and for philosophy, of the latest attempts to effect an indissoluble union between logic and mathematics in the light of the new mathematical theory.

Dedekind and Cantor are the mathematicians who definitively formulated this theory of aggregates, and Mr. Russell and Josiah Royce are typical representatives of those philosophers who have most diligently insisted upon the positive value of its implications for philosophy. The results of their logical investigations will accordingly occupy our attention throughout the remainder of our Study.

Mathematicians regard the work in connection with the theory of aggregates of such great importance that they have even given it a special label, namely, the tendency towards the 'arithmetization' of mathematics. "Arithmetization," says Mr. Hobson, "is the movement which has resulted in placing Analysis on a basis free from all notions derived from the idea of measurable quantity; the fractional, negative and irrational numbers being so defined that they depend upon the conception of integral number."³ For Mr. Russell, indeed, "all traditional pure mathematics can be derived from the natural numbers,"⁴ so that even the problems presented by space and time are, in one respect, at least, already present in, and solved by, the methods of pure arithmetic.⁵ Such, at all events, is the tendency, which is stated without discussion, indicated by the term 'arithmetization'. Incidentally, it is interesting to bear in mind that Leibniz also began his researches in mathematics from essentially this same standpoint.⁶

As the very name of the theory indicates, the most fundamental conception connected with it is that of an aggregate (*Menge, class, ensemble*). In the excellent *Encyclopédie des Sciences Mathématiques*⁷ an assemblage (or aggregate, or collection) is defined as a concept applicable to a number of objects, finite or infinite, "these objects being in general mathematical entities of the same nature, such as numbers, points on a line, points in a space of n dimensions, etc., etc." And is it not interesting to note that the very factors which Leibniz recognized as essential to the complete notion of any quantitative whole (such as an assemblage) are also regarded by the authors of the theory in question as constituting the elements of their conception? Namely, there is involved the single, discrete unit, plurality, and totality or unity.⁸

³ *Theory of Functions*, 1st ed., p. 2. Cf. also Shaw, *The Philosophy of Mathematics*, p. 23; and Russell, *Principles of Mathematics*, p. 259.

⁴ *Introd. to Mathematical Phil.*, p. 4.

⁵ *Prin. of Math.*, p. 259.

⁶ Husserl goes so far in this connection as to assert that Leibniz's *Ars Combinatoria* makes him "the spiritual father of the theory of aggregates," now to be studied. Cf. *Logische Untersuchungen*, I, p. 222.

⁷ Tome I, Vol. I, pp. 489-490.

⁸ *Op. cit.*, p. 493. Above, Ch. II.

Moreover, it may be well to emphasize the fact that the discrete objects of which, *e.g.*, an assemblage is composed, must be, from the point of view of the purpose at hand, of the same general nature, possessed of some continuous quality or 'logical parity'.⁹ This, again, was precisely Leibniz's meaning of the term 'quality'; *i.e.*, a logical determination of any sort.

Further specifications of the properties of an assemblage, or of assemblages in their relation to each other, are introduced by means of the notions associated with the terms cardinal number, ordinal number, and equivalence. Thus, two aggregates are said to be *equivalent* when the elements of the one can be placed in reciprocal, one-one correspondence with the elements of the other aggregate. And two aggregates, so equivalent, are said to have the same 'cardinal number'.¹⁰ As an example of such equivalent aggregates we may cite that of husbands and wives in countries where monogamy is practised. Again, an aggregate, finite or infinite, is said to be *enumerable*, if it be possible to establish a one-one correspondence between the elements of the aggregate and the natural numbers (1, 2, 3, . . .); otherwise it is non-enumerable.¹¹ An example of an enumerable aggregate is that of the proper fractions arranged in the usual order; of a non-enumerable aggregate, that of non-terminating decimal fractions arranged in the usual order.¹²

Even if the aggregate be non-enumerable, mathematicians are still able to distinguish in it some useful properties, of which the most interesting one for our study is that associated with the term 'continuum'. In order to understand the definition of this term, however, some preliminary definitions are necessary. First of all, a *segment* of an aggregate (viewed as a series) is any part *C* of the series, such that, if (*a*) and (*b*) are any two elements belonging to *C*, then every element between (*a*) and (*b*) also belongs to *C*. A segment *C*, such that if (*a*) belongs to *C* then every element that precedes or follows (*a*) also belongs to *C*, is called, re-

⁹ Hobson, *op. cit.*, pp. 1-2.

¹⁰ *Ibid.*, p. 8.

¹¹ *Encyclop.*, *loc. cit.*, p. 490.

¹² Huntington, *The Continuum*, pp. 39 and 42.

spectively, a lower or upper segment of the series.¹³ Secondly, such a segment is said to have a *limit* if in the series *S* there be a term which is the first after all the terms of *C*. In other words, a term (*x*) is the limit of the fundamental segment *C* if all the terms of *C* are inferior in rank to (*x*), and if every term of superior rank to all the terms of *C* is posterior to (*x*). The series is *perfect* if all its fundamental segments have limits and if all its terms are limits of fundamental segments. Lastly, a series is said to be *compact*, if between any two of its terms there exists at least one other term. An aggregate, now, is said to be a continuous series "when (1) every segment, upper or lower, has a limit, and the series has a first and last term; (2) an enumerable, compact series is contained in it in such a way that there are terms of this latter series between any two terms of the original series."¹⁴ The interesting point in this rather technical definition is the fact that, for the mathematician, continuity is apparently dependent upon discreteness.¹⁵ Thus, examples of a continuum so defined are the totality of real numbers, and the totality of points on a line. Both the totality of real numbers and a line, then, are defined as an infinite series of elements to which the ordinary cardinal number does not apply, and for which Cantor invented a new type of 'transfinite' numbers, applicable especially to infinite aggregates.¹⁶

This mathematically defined continuum must not be confused, however, with the logical factor of continuity implicit, according to Leibniz, in every quantitative determination of things. For, as a matter of fact, the continuum itself is a good example of the presence of both continuity and discreteness in the quantitative whole. The latter factor is present in the form of the discrete elements—numbers, points, or what not—which compose the continuum, while continuity manifests itself, of course, in the shape of the 'norm' which specifies whether any given element belongs, or does not belong, to the particular type of continuum in question.¹⁷ And this point it will be useful to bear in mind in the sequel.

¹³ Huntington, *op. cit.*, p. 37.

¹⁴ Russell, after Whitehead, *Prin. of Math.*, p. 299, note.

¹⁵ Russell, *loc. cit.*

¹⁶ Huntington, *op. cit.*, p. 75.

¹⁷ Hobson, *op. cit.*, Ch. III.

Many problems still remain unsolved as to the more detailed characterization of transfinite or infinite cardinal numbers, and indeed such authorities as Dr. Hobson have questioned the validity of the whole conception.¹⁸ But if the conception of infinite aggregates be admitted tentatively, it will be found to involve another property, quite different from the corresponding one associated with ordinary aggregates, and which, in the theory we are studying, serves as the distinctive quality or positive definition (to be substituted for the negative definition of mere endlessness) of mathematical infinity. Namely, in the case of finite aggregates, a whole is greater than any of its parts, but in the case of infinite aggregates, the part is said to be (*not equal*, but) equivalent—in the sense of equivalence above defined—to the whole. For example, a reciprocal one-one correspondence may be set up between the infinite series of positive integers and that portion of the series comprised of all the even numbers (also an infinite series), so that, according to the definition, the two series or aggregates are equivalent.¹⁹

Now the terms 'whole' and 'part' mutually imply each other, and both the precise nature of any whole, and the relation between whole and parts, plainly depend upon the categories under which the kind of objects with which we are dealing is properly subsumed. In the case of objects under the category of quantity, according to Leibniz, no whole in the sense of a sum of determinate parts exists objectively in its own right; for every sum may be viewed as an element in some larger whole, and *vice versa*, every element may be viewed as a sum of lesser elements. Nevertheless the 'whole' and 'part' here in question are equivalent, in the sense that the terms of the one may be related to the terms of the other in such a way that they may be viewed under a common aspect. Both the whole and the part, if it were possible to present them directly as definite sums, as finite sums are presented, would *ex hypothesi* be found to contain an infinite series of elements; and, moreover, these series would be alike, or equivalent, in that they both are composed of positive numbers. And because of the

¹⁸ *Loc. cit.*

¹⁹ Huntington, *op. cit.*, p. 6; *Encyclop.*, I, p. 493.

endlessness of both series, it is assumed that to each member of the series of positive numbers there would correspond a member of the series of even (positive) numbers. In this sense, no doubt, the latter series might be said to 'represent' the former series, for '2' would represent '1', '4', '2,' and so on as far as we cared to pursue the correspondence.

But we must pursue no further, at present, this interesting phase of the logic of the theory of aggregates. That the preceding remarks have a certain relevancy with regard to the applications which Royce and Mr. Russell seek to make of this mathematical definition of infinity to the solution of the properly philosophical problems concerning infinity, however, we shall discover as we proceed.

The next notion to be introduced is that of *order*. So far no mention has been made explicitly, say, of the nature or function of ordinal numbers, but mathematicians find that the most convenient way of dealing with certain problems of the continuum and infinite aggregates generally is by means of this conception.

Given an aggregate of elements (which, it may be recalled, may be constituted by any kind of mathematical beings, numbers, points, functions, etc., etc.), this aggregate is said to be *ordered*, or, in more precise language, simply ordered, if a norm or convention may be established which makes possible the arrangement of these elements in a certain relative rank or succession. That is to say, each element must be recognized as possessing a certain logical 'position,' in virtue of which it is known as regards any two elements, which of them has the lower, and which the higher rank. One element is said to 'precede,' or to be 'less than,' or to be 'below,' or 'to the left of,' etc., any other element of higher rank than itself.²⁰ An easy extension of this idea of simply ordered aggregates is that in which the elements have not one, but many (say n), relations to each other and then the aggregate is said to be multiply ordered.²¹ An example of simply ordered aggregates is furnished

²⁰ Hobson, *op. cit.*, pp. 1-2; *Encyclopédie*, I, p. 498; Young, *Fundamental Concepts of Geometry and Algebra*, pp. 67-9.

²¹ *Encyclop.*, p. 498, note.

by the series of real numbers; of multiply ordered aggregates, by the coördinates of space of n dimensions.

When consideration is turned to two or more ordered aggregates we discover further interesting characteristics. Two such aggregates are said to be *similar* if there exists between their elements a reciprocal one-one correspondence, such that, (a) and (b) being two elements of the one and (a') and (b') the corresponding elements of the other aggregate, the condition that (a) 'precede' (b)—symbolized by $a < b$ —involves that (a') precede (b')— $a' < b'$.²² Secondly, aggregates thus similar to a same aggregate are said to have a common property called a 'type of order,' from which is derived the notion of ordinal number, just as the notion of cardinal number is derived from that of equivalent aggregates.

With the discussions among mathematicians as to the relative primacy of cardinal or ordinal numbers, or as to the 'empirical' or 'logical' origin of number, we need not at present concern ourselves. Likewise the interesting details and further qualifications associated with the application of these various notions to the special problems of arithmetic, analysis and geometry, must be passed over as outside the scope of the present Study. The meaning which has become attached to the mathematical concept of order, however, is so important, and it has exercised so profound an influence in connection with the later development of the principles of mathematical logic, that it merits some special attention. Mathematicians have found that the notion of types of order, as briefly explained above, is an essential element in many important subdivisions of their science. The real number system of algebra; the integers and rational numbers of arithmetic; the aggregate of points on a line, of lines through a point, and the dimensionality of geometry; the conception of a limit which is so important in higher mathematics—all either depend upon, or to some extent involve, the notion of order.²³ Inevitably, therefore, any logical study of the science must, as a part of its task, concern itself with this very general notion; and, indeed, it has been employed by the mathemat-

²² *Encyclop.*, p. 498.

²³ Young, *op. cit.*, Ch. VII; Russell, *Introd. to Math. Phil.*, Ch. IV; *Encyclop.*, pp. 498–531.

ical logicians to constitute a mode of passage from mathematics to (what they call) logic.

Now the difference between Leibniz's views on the relation of mathematics to logic and those of the moderns is well illustrated by their respective attitudes towards the notion of quantity, and of the relative significance of this notion and that of order. "In former days," Mr. Russell explains, "it was supposed (and philosophers are still apt to suppose) that quantity was the fundamental notion of mathematics."²⁴ Leibniz, it will be remembered, said that algebra, the fundamental subdivision of mathematics, was the "general science of quantity." And not to mention Hegel, there are several contemporary logicians who, in general, maintain some such view as that imputed to them by Mr. Russell.²⁵ "But nowadays," our authority continues, "quantity is banished altogether, except from one little corner of geometry, while order more and more reigns supreme. The investigation of different kinds of series and their relations is now a very large part of mathematics, and it has been found that this investigation can be conducted without any reference to quantity, and for the most part, without any reference to number. All types of series are capable of formal definition, and their properties can be deduced from the principles of symbolic [*i.e.*, mathematical] logic. . . ."²⁶

Boole had already expressed a similar view in his *Laws of Thought*; in closely identifying mathematics and logic he was led to deny that the former science concerns itself solely, or even principally, with number and quantity.²⁷ Royce, too, scoffs at the "absurdity" of the opinion, "still often entertained by certain philosophical students, that 'mathematics is the science of quantity'."²⁸ Most clearly of all, perhaps, Couturat explains how, as a result of greater accuracy and precision, the relations between mathematics and logic have come to be envisaged under a new aspect, until it now appears "that the sciences of number and magnitude are not primordial, but rest upon doctrines of a logical

²⁴ *Mysticism and Logic*, p. 91.

²⁵ Bradley, *Principles of Logic*; Bosanquet, *Logic*.

²⁶ Russell, *op. cit.*, p. 91. Below, Ch. VI.

²⁷ *Op. cit.*, p. 12.

²⁸ *Encyclop. of the Philosophical Sciences*, p. 105.

rather than a mathematical character, and on notions which are in no sense *quantitative*.”²⁹

This viewpoint evidently demands careful examination, since it brings up some very important issues. If, in the first place, those who concern themselves with the logical principles of mathematical science reject the notion that this science rests upon the category of quantity, it is natural to believe that they have very good grounds for so doing. On the other hand, we must also ask, what is their own interpretation of the meaning of this word, ‘quantity’? This, indeed, is a problem which logicians have always shared with mathematicians, and we have seen that Leibniz gave it some attention and came to certain (already noted) conclusions. We feel it as a serious deficiency, then, when we discover that mathematical logicians have paid so little attention to the attempts of other logicians (*e.g.*, Kant, Hegel, Bosanquet) to analyze the content of this category. Merely to scoff at the “absurdity” of restricting mathematics to the domain of quantity, as Royce does, solves no logical problems. We have a right to demand of him as a logician what his own interpretation of this category is, and should also expect a thorough criticism of the view he rejects. This demand becomes more pressing when he proceeds to assert that “such concepts as number, quantity, space, [and] time . . . have been dealt with by the pure philosophers far too summarily and superficially.”³⁰ No convincing evidence is given, however, either by Royce or his colleagues, of the summary treatment accorded to this matter by “the pure philosophers.” On the contrary, Royce contents himself, in common with Mr. Russell and Couturat, with an independent analysis of the logical presuppositions of mathematics. And this analysis, be it noted, depends upon an uncritical acceptance (so far as Royce is concerned) of the view that the ideas associated with the notion of order constitute the most fundamental portions of the science.

To understand Royce’s conception of order and quantity, and of the relation between them, a preliminary exposition of his treat-

²⁹ *Princ. des Math.*, pp. 2, 98–99. Cf. also Johnson, *Logic*, I, p. xxvi; Husserl, *Logische Untersuchungen*, p. 252; Cassirer, *Substance and Function*, Eng. transl., Chaps. I and II.

³⁰ *Science*, Vol. XX, p. 455.

ment of the "Problem of the One and the Many," in his Gifford Lectures on *The World and The Individual*,³¹ may prove helpful. Philosophical matters aside, the solution of the problem which he gives there is simply an interpretation of the mathematical account of the nature and properties of an infinite aggregate, essentially as already summarized. And the conclusion which Royce reaches in the Essay is that a number-system, such as the infinite series of real numbers, is, logically speaking, a "formally ordered collection of ideal objects."³²

The important word to emphasize here is 'order'. Once we have determined the order in which any element in the series shall stand to any other element, a means of generating the series—even if it be infinite—is thereby acquired. Now the philosophical moral of this interpretation is, that the order, the law, which prescribes the nature of the series, represents the One, while the elements thus generated represent the Many of the series in question.³³ In our own words, the One represents the factor of continuity, while the Many stands for the complementary factor of discreteness. "But all order in the world of space, of time, of quantity, or of morals," Royce continues, "is order because it presents to us systems of facts that may be viewed as having a first, a second, a third constituent, or some higher order." More particularly, "the deepest essence of the number series lies not in its power to aid us in finding *how many* units there are . . . , but in its expression that something is first, and something next. . . ."³⁴

Leaving aside for later consideration the philosophical aspects of the matter, we learn one thing from this quotation. Namely, that for Royce, order, as interpreted by mathematicians, is a notion applicable to all spheres of thought, and is by no means restricted to the field of quantitative mathematics. As for quantity, it represents by comparison only a subsidiary interest; it merely answers the question, 'How many'. From this high ground Hegel's analysis of the quantitative infinite appears "wholly inadequate to the

³¹ *Op. cit.*, I, "Supplementary Essay," pp. 482-588.

³² *Ibid.*, p. 517.

³³ *Ibid.*, pp. 526 ff.

³⁴ *Ibid.*, pp. 535-536.

complexity of our problem and rather hinders than aids its final expression.”³⁵

In this view, ordinal number, or, as Royce prefers to say, order in general, is entirely separated from the conception of cardinal number or sum, and is considered the more fundamental notion. It is, indeed, endowed with a philosophical significance which it would be difficult to surpass. Not only man, but “every rational being” must, precisely in so far as he is rational, be capable of distinguishing this ordinal aspect of things! Quantity, on the other hand, seems to be identified with what is considered as the far less important cardinal aspect of number. “Quantities,” Royce shortly says, “are objects, either physical or ideal, that fall into series by virtue of relations of the nature of *greater* and *less*” (or equality).³⁶ So we come back once more to the final, if somewhat abrupt conclusion that “what makes deductive inference in the realm of quantity [as elsewhere] possible depends solely upon the ordinal properties of this realm.”³⁷

From the point of view of mathematics, and quite apart from the philosophical interests involved, there are two exceedingly important issues here at stake. The first one is that of the relation of cardinal number to ordinal number; the second, that of order to quantity.

On the first issue modern mathematicians are very much divided.³⁸ Mr. Russell’s spirit of caution in this regard is therefore one which should have commended itself to Royce. In general, however, Mr. Russell does incline to the view that cardinal number is more naturally ‘first for us’.³⁹ In the light of such various interpretations, then, Royce’s dogmatic assurance strikes one as a bit hasty, especially since he neglects wholly to discuss adequately other interpretations than his own. What right had he as a philosopher, interested in mathematics principally for its logical import, thus to pass over in silence the painstaking investigations of other philosophers on this very point? For example,

³⁵ *Op. cit.*, p. 527, note.

³⁶ *Encyclop. of Philos.*, p. 115.

³⁷ *Ibid.*, p. 116.

³⁸ *Encyclop. des Sc. Math.*, *loc. cit.*; Hobson, *op. cit.*, Ch. III.

³⁹ *Prin. of Math.*, *etc.*, *passim*.

Hegel, in his *Larger Logic*,⁴⁰ devotes considerable study to the relation between extensive and intensive *quantum*—i.e., cardinal and ordinal number—with the result that, for him, these terms represent two ultimately inseparable factors of the complete concept of number. As factors, they may, indeed, be studied each for itself, when the purposes of the mathematician require it; but in the end, one such aspect always logically involves the other. Neither is substantial corroboration lacking of this thesis, on the part of contemporary mathematicians. In his *Domain of Natural Science*, for example, Dr. Hobson maintains essentially the same position. “Cardinal and ordinal numbers,” he explains, “are distinguished from one another in that their descriptive functions are different.”⁴¹ To attempt to determine which of the two originated first, or is logically prior, therefore, is a misdirection of effort analogous to the discussion concerning the primacy of the universal and the particular—or the hen and the egg.

The burden of proof, plainly, falls upon Royce. Order will be, for him, the supreme mathematical concept, with tremendous logical and philosophical implications. Not only, then, must he show such a view as that of Hegel, or that of Mr. Russell, with regard to cardinal numbers, to be in error; he must also demonstrate how cardinal number may be derived from ordinal number. Yet he has carried out neither task; he has not even mentioned as a possibility Hegel's or Dr. Hobson's view; and so far is his work from showing the relative unimportance of cardinals that, as we shall contend throughout, he actually employs the ideas connected with this type of number from the very beginning.

On the second issue, that of the relation of order to quantity, we have seen that Royce and Mr. Russell are in substantial agreement. The latter, indeed, neglects, if possible even more completely than does the former, any criticism of the views of other students of logic. As a substitute for such criticism he contents himself with the remark that prior to the work of Dedekind and Cantor “the view prevailed that number and quantity were *the* objects of mathematical investigation.”⁴² He then refers to the tendency of

⁴⁰ *Werke*, Vols. 3-4, pp. 242-253.

⁴¹ *Op. cit.*, pp. 104-105.

⁴² *Prin. of Math.*, p. 157.

'arithmetization' and asserts that careful analysis of this movement shows how all the material employed in modern analysis can be defined in purely logical terms; that is, in entire independence of any empirical data—which constitutes rather the material of applied mathematics—or of the mention of any such notion as that of quantity. Number and quantity, then, are entirely separate notions, and most theorems relating to quantity "can be generalized so as to become theorems concerning order."⁴³

In connection with this important conclusion we may well recall Leibniz's alternative solution of a very similar problem. For him, as we have seen, quantity does indeed reveal itself as indissolubly bound to the domain of perceptual experience. Space, time, and number, nevertheless, are 'ideal,' not metaphysically real things. Their ideality, perhaps we may say, comes not from their entire apartness from experience, but from the fact that they transcend indefinitely while at the same time always having reference to, and their root in, that particular type of experience which is limited to the perceptually apprehended. Thus we may surmise that order, in the sense of the word employed by modern mathematicians, would have approved itself to Leibniz as an important quantitative concept, combining within itself, and serving as the generalized expression for various special mathematical relationships or attributes.⁴⁴ These contemporary thinkers,—Royce and Mr. Russell—however, propose another interpretation. The fact that the generalized notion of order is involved in many mathematical constructions, which as such admittedly transcend the domain of any merely empirical experience, they interpret as proof that these mathematical conceptions and constructions have no essential connection with, but only an incidental application to, such experience. In other words, the fact of transcendence is unconsciously confused with the question as to origin. This confusion manifests itself, in that the universally admitted fact comes uncritically to be regarded as sufficient evidence for holding that such conceptions have their origin in an entirely separate realm,

⁴³ *Op. cit.*, p. 158.

⁴⁴ *Cf.*, in this connection, William Benjamin Smith's *The Philosophy of Mathematics*.

qualified by Mr. Russell as 'logical,' and by Royce as 'ideal,' to indicate its essential independence of the supposedly barely empirical sphere of physical existence.

The import of this point of view for logic and mathematics we shall have to consider at length in the following chapters. In the meantime we may briefly notice the more important fruits, for philosophy, of this doctrine.

Its philosophical significance depends upon the fact that—as explained earlier in the chapter—the new mathematical definition of infinity may be formulated in terms of the concept of order. Both Royce and Mr. Russell proceed to make capital of this definition for the solution of the philosophical problem of the infinite regress, asserting that the mathematical infinite, so conceived in purely 'logical' or 'ideal' terms, is freed from the reproach of mere endlessness formerly leveled by philosophers against quantitative conceptions in general. According to Mr. Russell, for example, the problems of infinity, the infinitesimal, and continuity, for so long insoluble puzzles both to mathematicians and philosophers, are all definitely solved by means of the concept of order, and are not properly quantitative concepts at all.⁴⁵

Now if this claim were valid, Realism would have achieved a signal victory. The realistic philosopher would be able to unloose the shackles of finitude and discreteness, to which reference has previously been made, and formerly inevitably implied in his insistence on taking the immediately given as alone real. For infinity now appears to be 'given'—if not empirically then—in an immediate intellectual intuition or apprehension of purely 'logical'—*i.e.*, non-empirical—data, while continuity, for its part, seems to be defined in terms of its enemy, discreteness itself.

We have already observed, however, that as a matter of fact it is not continuity in the Leibnizian sense, but only the quite distinct mathematical conception of the continuum which may be so defined. It is obvious at once, therefore, that Mr. Russell's endeavors to make the latter conception do service for the former can only result, everywhere, in confusion and error. And finally,

⁴⁵ *Prin. of Math.*, pp. 188–196.

although we do not presume to evaluate, for the scientist, Cantor's famous definition of the mathematical infinite, we may yet venture to indicate what we hold to be one of its principal defects from the point of view of the philosophers whose doctrines we are studying. Namely, both Josiah Royce and Mr. Russell emphasize as of supreme significance the character of immediate givenness which, they assert, the Cantorian infinite plainly manifests. It is essential for the purposes both of Mr. Russell, who values as alone real the immediately apprehensible, and of Josiah Royce, who demands an entirely immanent criterion of reality or the Self, that the infinite regress of mere endlessness be definitively arrested.⁴⁶ But regardless of the *claims* of scientist or philosopher, can we actually be assured that the new infinite does indeed manifest the character of givenness attributed to it? One can see, perhaps, that *if* these self-representative numerical correspondences or series *were* completed, according to the law of their generation, then, assuming that law to hold in the end as at the beginning, the same correspondences would persist throughout. But, as a matter of fact, no series of the type in question is ever given as complete; there are always terms beyond those recorded, as indeed the admission of endlessness,—as a 'negative,' but nevertheless necessary element—plainly indicates. In short, the mathematical infinite is no more given as a completed series of numbers than is the actual value of π , or of any transcendent number, so-called. In the one case as in the other, what *is* given is a method of generation of terms *ex hypothesi* without end and therefore *not* given. And this method or law of the generation of the infinite aggregate is on precisely the same logical level as any other mathematical law of series, *e.g.*, the law of the generation of the series of whole numbers. In other words, no more than other mathematical examples of mere endlessness does this new one manifest more than the fundamental incompleteness, inherent, as Leibniz recognized, throughout the domain of quantity—an incompleteness fatal to its philosophical value as an actualization of the true or self-determinate Infinite.

⁴⁶ Cf. *The World and the Individual*, I, "Supplementary Essay."

The outcome of this chapter has been to show (1) that for the mathematical logicians the science of pure mathematics depends to a great extent upon the so-called logical concept of order, rather than upon the 'traditional' category of quantity; but (2) that their views on this point are erroneous, at least as to the relation of order to quantity—that though we grant them the right to define their terms as they please, these thinkers actually and in principle employ throughout, without ever transcending, the very factors or 'moments' which, traditional logic has vigorously insisted, comprise the content of this category. (3) Lastly, it seems safe to conclude that no such mathematical concepts as those of the continuum and the infinite of the present Theory of Aggregates, however useful and significant they may be in their proper domain, and for the time being, can render direct service in the field of philosophy. Only a confusion of categories can result from such a misapplication of terms, as, *e.g.*, of the mathematical continuum for logical continuity.

In view of so many misinterpretations of the real doctrines of traditional logic, and in the light of what we must for the present consider as so many false conceptions with regard to the presuppositions of mathematical science, it may seem that there is little ground for pursuing our subject further. But even errors are illuminating, and in this respect we shall find matter of interest in connection with the precise manner in which the mathematical logicians seek, by means of the concept of order and kindred notions, to derive the science of mathematics from the science of logic.

CHAPTER IV.

THE GENERAL CHARACTER OF MATHEMATICAL LOGIC.

The mathematical logicians' assertion that order is a 'logical' concept naturally presupposes a certain conception of the nature and function of logic itself, as well as a definite notion of the relation of logic to mathematics. And such presuppositions, of course, involve philosophical issues of great and far-reaching significance. The present chapter contains a brief review and comparison of the respective views of Mr. Russell and Josiah Royce on these and correlative matters. We shall not, however, attempt to deal with the philosophical speculations of these thinkers at large, but shall restrict the inquiry to those portions of their thought which have to do with our more limited subject.

"By the name mathematical logic," says Mr. Russell, "I will denote any logical theory whose object is the analysis and deduction of arithmetic and geometry by means of concepts [such as order] which belong evidently to logic."¹ The implied converse of this definition is that the principles of mathematical logic are simply the result of a more complete analysis of the fundamental mathematical concepts. There is therefore little importance in distinguishing the domains of the two sciences, mathematics and logic. The former is just a continuation of the latter, or *vice versa*, the latter is a still further generalization of the former field. "Starting with premises which would be universally admitted to belong to logic, and arriving at results which as obviously belong to mathematics, we find that there is no point at which a sharp line can be drawn with logic to the left and mathematics to the right."² Such a science it is appropriate to call, neither mathematics in the narrower sense, nor logic in the usual sense, but mathematical logic in a sense which somehow is to include them both.

¹ "The Philosophical Importance of Mathematical Logic," *Monist*, Vol. XXIII, p. 481.

² Russell, *Introd. to Math. Phil.*, p. 94.

As a direct result of this close interconnection now established to the satisfaction of certain thinkers between the two fields of logic and mathematics, the latter science becomes a study apart, possessing characteristics which clearly distinguish it from the other exact sciences, so-called. Mr. Johnson, for instance, points out that "mathematics . . . is nothing but an extension of logical formulæ, introducing none but purely logical factors; while every other science borrows its material from experiential sources and can only use logical principles when or after such material is supplied."³ Royce adopts a somewhat similar attitude. To his mind, the philosopher and the mathematician are equally engaged, if from different angles, "in the consideration of an ideal realm."⁴ In this regard philosophy and mathematics are to be contrasted with the sciences which have to do with "the description of external facts," *e.g.*, physics and astronomy.⁵ In short, Royce appeals to "the spirit which Descartes and Leibniz illustrated."⁶

That is to say, Royce, and, in common with him, Mr. Russell, Couturat, Husserl, and others, maintain the view that the fundamental conceptions both of logic and of mathematics—which, indeed, they in a sense hold in common—are the work alone of pure or abstract thought. Thought, that is, as a separate faculty of the mind, is considered capable of abstracting from all contact with the *ex hypothesi* separate world of sense-perception, and of moving in an ideal realm of its own, where it creates or discovers, *a priori*, its own objects, and the most general laws governing their theoretical (logical and mathematical) consideration.

Perhaps enough has been said to make it evident that for the mathematical logician the ideal for logic to pursue is that of the utmost generality and abstractness. One might hastily assume from this that logical propositions therefore properly refer only to certain very abstract properties of the world as a whole. Such, indeed, Mr. Russell for his part believes, is the doctrine of the 'traditional' school in logic and philosophy. But if we surrender as self-contradictory—as this authority maintains that we must⁷—

³ *Logic*, I, p. xxiv.

⁴ "The Sciences of the Ideal," *Science*, N. S., Vol. XX, p. 453.

⁵ *Loc. cit.*

⁶ *Ibid.*, p. 449.

⁷ Cf. *Scientific Method in Philosophy*, p. 39; *Mysticism and Logic*, p. 110.

the notion of a unitary whole, then only one alternative remains: namely, that there exist or are possible, only particular things, single objects, or independent entities. And while to this thinker's mind there is no basis, empirical or otherwise, for believing in the existence of one interrelated universe, he maintains that examples (or, more strictly, evidences of the presence) of such elementary particulars are to be discovered by observation, *e.g.*, sense-perception, actually existing in the external physical universe. This, then, is the grain of truth in empiricism. But a complete account of our experience cannot be rendered in terms of empiricism. It could be so rendered only if philosophical truth were dependent upon "special and accidental facts," such as "the course of history, or the convolutions of the brain, or the eyes of shell-fish"; in short, upon merely perceptual evidence.⁸ But not only is it fairly evident that there are independent self-contained particulars in the world of sense-perception: it is apparent also, so the argument continues, that abstract thought is capable of constructing, quite apart from perceptual experience, an infinity of other particulars, such as the square root of minus one, or an electron. That being so, we may say shortly that the corresponding quality of pure logical propositions is that they must be *a priori*,—free from the limitations of a too narrow empirical outlook. That is, the propositions of logic "are concerned with such properties of all things as do not depend upon the accidental nature of the things that there happen to be, but are true of every possible world, independently of such facts as can be discovered by our senses."⁹

The philosophical moral of this conception of things, after what has been said, may be summed up in a word. "The philosophy which I wish to advocate," says Mr. Russell, "may be called logical atomism or absolute pluralism, because while maintaining that there are many things, it denies that there is a whole composed of those things."¹⁰ In other words, we have to do with the extreme opposite of a 'block-universe' theory. Of this pluralistic philosophy, ethics and religion and the consideration of values

⁸ *Mysticism and Logic*, p. 111.

⁹ *Loc. cit.*

¹⁰ *Ibid.*, pp. 111 and 113.

constitute no essential part. Philosophy, like mathematical logic, or rather, in common with it, is to be guided solely by the scientific ideals of exactness, rigor, and the utmost generality. In sum, philosophy is to become 'scientific' both in method and in function; in this respect, indeed, it is to exceed in generality any experiential science such as physics. And it appears that to be 'scientific,' in this sense of the term, means, in the first instance, at least, to employ the method of analysis, which is in fact none other than "the essence of philosophy."¹¹

The real, then,—for we cannot say reality—is the existential particular. In words which vividly recall Leibniz's youthful views, Mr. Russell explains that as a result of a complete logical analysis (supposing that it could be carried out, which admittedly is doubtful), "there will be one word and no more for every simple object, and everything that is not simple will be expressed by a combination of words, by a combination derived, of course, from the words for the simple things that enter in, one word for each simple component."¹²

But while the particulars, which form the subject-terms of propositions, are so very real and objective, our ordinary matter-of-fact propositions themselves are merely the expression of the way in which we individually know the world, and are therefore unavoidably subjective and psychological. For Mr. Russell, this means that a distinction must be made between strictly logical, or ideally envisaged premises, such as those of mathematical logic, and the premises of a theory of knowledge. By an hypothetically complete analysis of a given material we would arrive, theoretically, at logically primitive propositions, and such propositions would form the necessary and objective basis of a perfectly rigorous logical system. But such propositions are not premises for the theory of knowledge. "It is important to realize the difference between that from which your knowledge is, in fact, derived, and that from which, if you already had complete knowledge, you would deduce it."¹³ The logician's premise is one ideally pre-

¹¹ *Op. cit.*, p. III.

¹² "The Philosophy of Logical Atomism," *Monist*, Vol. XXVIII, p. 520.

¹³ *Ibid.*, p. 498.

supposing complete knowledge, therefore, and while possessing "great deductive power, great cogency and exactitude," is arrived at only after a long and careful preliminary analysis. Indeed, it is not at all certain that such an analysis can ever be completed. It is only an ideal goal, and practically we can never be certain of having attained it.

However that may be, everybody would agree as to the validity of the conclusions deduced from such ideal premises. Quite different, however, is the premise that knowledge actually starts from. "When you are talking of the premise for theory of knowledge, you are not talking of anything objective, but of something that will vary from man to man, because the premises of one man's theory of knowledge will not be the same as those of another man's."¹⁴ Hence it comes about that there will be, in practice, as many 'logics' as there are different sets of subjectively asserted premises, *i.e.*, an infinite number, and conclusions correspondingly different as to content, but equally valid as to form. In other words, an absolutely inescapable subjectivity is the heavy price which the mathematical logician finds himself forced to pay, as a recompense for his complete detachment from the limitations of perceptual experience. And it would seem to follow from this that mathematical logic can only tell us how we *would* think, if certain impossible conditions were actually given or granted. Instead of effecting a satisfactory union between mathematics and logic, the logisticians seem, indeed, rather to have relinquished both of these sciences for the sake of hollow abstractions and hypo-statized conceptions!

Incidentally, and as a corollary to this important, though highly questionable distinction between logic and epistemology, between the products of an ideally complete analysis, and the actual starting-points of such knowledge as we poor mortals possess, a word of caution, which Mr. Russell kindly offers for our guidance at this point, verges on the comical. Namely, he warns, we must not make Kant's mistake. We must not assume, with that philosopher, "that the universality [*i.e.*, generality] of *a priori* truths comes from their property of expressing properties of the mind."

¹⁴ *Op. cit.*, p. 499.

For this assumption would imply, according to the critic, that we can never know the objects of the external world except in their relation to mind, and hence as mere subjective phenomena. Moreover, how could we be sure on this hypothesis, that two minds would see the world alike?¹⁵ So that "as soon as we take into account the consequences of Kant's hypothesis, it becomes evident that general and *a priori* truths must have the same independence of mind that the particular facts of the world possess." The only possible alternative, Mr. Russell argues, would be to suppose that these general truths express only psychological facts, of which we could not be assured of the permanence from moment to moment, nor from person to person. Neither could we carry on a valid reasoning process, since we would always be the dupes of our own ideas. "Logic and mathematics force us, then, to admit a kind of realism in the scholastic sense, that is to say, to admit that there is a world of universals and of truths which do not bear directly on such and such a particular existence."¹⁶ Thus the universals of logic 'subsist' in an objective realm of their own, the realm of the merely hypothetical, the barely possible, and the purely *a priori*. Like the objects of sense-perception, these universals are independent of the empirical mind; and just as the objects of the physical order are immediately intuited in perception by the senses, so the objects of the logical order are intuited by an immediate intellectual intuition of pure thought. Consequently we may expect to find that the detailed content of these two worlds possesses "an analogous structure; [that] the complex propositions of the one decompose themselves into constituent parts, just as the corporeal aggregates of the other divide themselves into elements."¹⁷

Mr. Russell's objection to Kant's philosophy as involving reference to the nature of the mind, verges, we say, on the comical. For as clearly as may be, his very objections are based on a certain conception of mind which somewhat resembles what Kant called 'the Empirical Ego'. But since Kant himself always insisted on the inadequacy of this view of the mind as a kind of material

¹⁵ *Prin. of Math.*, p. 491.

¹⁶ *Mysticism and Logic*, pp. 91-92.

¹⁷ *Brunschvicg, op. cit.*, p. 392.

entity—on a par with the entities of the physical universe—as a basis for the solution of philosophical problems, the critic has merely succeeded in proving himself guilty of a deplorable error in interpreting, and not at all in refuting the Kantian philosophy of mind.

Now while Royce—to turn for a moment to his views on these issues—assigns to the conceptions constituting the world of pure logic the same characteristics as does Mr. Russell, namely, hypothetical, *a priori* possibility, it is interesting to observe that he orients this world differently with respect to the knowing subject. The latter thinker *finds* his self-evident truths; thought, merely by lucky accident, discovers their presence in the purely 'objective' realm where they subsist eternally. As if restrained by its taint of subjectivity, thought is passive at this stage. But with Royce on the contrary, it is in the active process of thinking, itself, that we 'construct,' 'create,' 'invent,' or 'find' "the pure forms of thought," "when we consider, not the physical, but the logical realm itself," studying it without regard to the physical world.¹⁸ Here the "logical realm," instead of subsisting wholly apart from the mind, is, precisely, "*meine Gedankenwelt*,"—and nothing else.¹⁹ Thus essentially the same realm which is for Mr. Russell quite 'without' the mind is for Royce wholly 'within' it; in fine we seem to have forced upon us a choice between two absolutely opposing views.

But after all, perhaps the most important element in this conception of an abstract logic is common to both thinkers; namely the naïve, uncritical postulation of an external, *i.e.*, spatial, relation between the world of thought and the physical universe. Thus Royce speaks of "the philosophical problem as to how and why the given facts of physical experience conform, as nearly as they do, to the forms of our thought,"²⁰ and Mr. Russell of the general truths of logic, which are "wholly independent of the accidental facts of the existent world."²¹

¹⁸ *Encyclop. of Phil.*, p. 93.

¹⁹ This phrase Royce borrows from Dedekind. Cf. *World and Individual*, I, "Supplementary Essay," pp. 511 ff.

²⁰ *Encyclop. of Phil.*, p. 93.

²¹ *Scientific Method in Phil.*, p. 57.

So much for certain recognized similarities between the theories of these two thinkers. As for the differences, Professor Lewis points out that the method which Royce employs is distinguished from the method of *Principia Mathematica* "by the attempt to substitute *selection within an initial order* for analysis (definition by previous ideas) of newly introduced terms."²² What this difference of method ultimately amounts to we shall learn in the sequel when we come to study the more specific details connected therewith. As for the "initial order" itself, Royce simply 'postulates,' *a priori*, the existence of a "system Σ ," consisting of an infinity of "simple and homogeneous elements,"²³ whose sole function it is to enable one to ensure the presence, and to formulate the properties of certain fundamental 'logical' relations.

These relations and their *a priori* properties, implicitly contained in the 'system,' or initial order, constitute for Royce the starting-point of his method. They thus assume the place, for him, which is occupied in Mr. Russell's procedure by the set of notions resulting from his analytical consideration of certain mathematical conceptions.

The function, then, of both of these methods—the method of analysis of a given material and direct intuition, and the method of *a priori* postulation and arbitrary selection—is to establish the general propositions which are to serve as the premises for deductive inference. But what of induction? Surely it is possible to make generalizations from observation and study of instances found in experience, which generalizations may then be used as valid, objectively grounded premises, from which to deduce new consequences. And is not such a premise more certain than one that only seems so to any individual or group of individuals? And is it not a fact that all of the actual, constructive advances, even in the mathematical sciences, such, *e.g.*, as the very important Theory of Aggregates itself, have resulted from critical reëxamination of existing, specifically mathematical assumptions, problems and postulates, rather than from such abstract, *a priori* constructions as those of mathematical logic?

²² *Survey of Symbolic Logic*, p. 363.

²³ *Trans. Amer. Math. Soc.*, Vol. VI, p. 362.

No, such empirical methods have no practicable warrant, says Mr. Russell. "You can never arrive at a general proposition from particular premises alone."²⁴ Consider, for example, the general proposition 'All men are mortal'. It is sometimes supposed that we arrive at this proposition by the process of 'complete induction,' *i.e.*, by actually enumerating 'A is mortal,' 'B is mortal,' and so on, until all men are included in the list. But the question is, how do you know when you have finished? Only by means of a presupposed general proposition, Mr. Russell replies, such as 'All men are among those I actually have enumerated'. Thus there must always be at least one general proposition in the premises along with the particular ones. From this it follows that knowledge of general propositions necessarily presupposes "primitive knowledge of general propositions (I mean by that, knowledge of general propositions which is not obtained by inference)."²⁵ So that "the problems of philosophy should be anterior to deduction . . . and here no method is available save intuition."²⁶

Of primitive knowledge, self-evidence, and general propositions we shall hear much in the sequel. Yet all that can be said about such things so far as their logical nature is concerned practically amounts to a confession of ignorance. There they are; and we cannot dispense with them as factors in deductive inference. And yet, on the other hand, Mr. Russell admits that he is not able to account for them rationally. So he concludes his discussion of inductive inference with this confession of failure: "It is an exceedingly difficult question, and one which I should very much like to see studied."²⁷ So far as induction is concerned, therefore, we find that Royce fails to consider it at all, while to Mr. Russell's mind it is of precisely the same linear character as is deduction, consisting in what we often call merely enumerative (and hence essentially defective) induction. That is, the process, as he understands it, simply passes from instance to instance, and the more instances are observed, the greater—supposedly—is the certainty of the inference.²⁸ But surely there is manifest here the same

²⁴ *Monist*, Vol. XXIX, p. 199.

²⁵ *Loc. cit.*

²⁶ *Leibniz*, p. 171.

²⁷ *Monist*, *op. cit.*, p. 201.

²⁸ *Problems of Philosophy*, Ch. VI.

lack of any principle of systematic coherence as, we are coming to see, characterizes all linear inference. In other words, there is no basis of rational necessity: one contrary instance obviously is sufficient to overthrow a conclusion based on such flimsy grounds. Indeed, the result of such a process is more properly called a mere generalization, and is therefore to be carefully distinguished from a genuine scientific induction. At all events, Mr. Russell shows by his dissatisfaction with his own account of induction that he is aware of the inefficacy of this general linear schema so far as inductive inference goes, though he is confessedly unable to propose a more satisfactory theory in its place. But the marvel is, that we shall find him persisting in the belief that deduction, any more than induction, can yield valuable results, so long as it is conceived of as consisting in the precise obverse of the same linear process.

CHAPTER V.

SOME FUNDAMENTAL CONCEPTIONS.

We now have before us, in general outline, the philosophical point of view—implying, as we have seen, an external relation of mind, *i.e.*, thought, to the empirical order of nature—and the more important ‘epistemological’ considerations as to the nature and function of thought, from which the methodology of the mathematical logicians directly derives. Before, however, we may consider ourselves adequately prepared to study the specific characteristics and results of the method of linear inference, as conceived and developed by Mr. Russell and Josiah Royce, respectively, we must observe briefly the detailed results of the analysis which forms the essential preliminary to this procedure. In other words, our immediate concern is to discover what are the fundamental conceptions, acquired by analysis from the concrete matter of mathematical science, upon which the present-day method of mathematical logic so completely depends.

Since knowledge, as such, obviously consists of propositions, the most elementary portion of mathematical logic must devote itself, according to Mr. Russell, to the analysis of such propositions, with a view to discovering their logical nature, and their function in inference.

All propositions, it appears, may be analyzed into certain ‘constituents’ and a certain ‘form’.¹ And—to repeat this often emphasized note once more—the fundamental error of traditional logic, so-called, consists precisely in the fact that it neglects entirely every form of proposition except only the subject-predicate form which Leibniz inherited from Aristotle. This latter doctrine, the critic urges, fails utterly to give mathematics its due, since the propositions of mathematics all have the form of relations. We may make the point clear by an illustration.

Take, for example, the proposition ‘Socrates is mortal’. Such a proposition contains two constituents, namely the subject-term

¹ *Introd. to Math. Phil.*, p. 200.

'Socrates' and the predicate 'mortal'. The former, strictly speaking, must always be a particular, in the sense already defined. As such, it stands by itself, in the proud independence of eternal self-identity and self-evidence. "The notion that a term can be modified arises from neglect to observe the eternal self-identity of all terms, and all logical concepts which alone form the constituents of propositions."² In the above proposition, 'mortal,' the predicate-constituent, is an adjective, corresponding to the concept 'mortality,' while 'is,' the verb, "merely indicates the subject-predicate form."³ This completes the analysis of one typical form of proposition. But even this elementary step has important consequences. For the assertion of the self-identity of the subject-term, as an independent unit, a logical element or atom, "destroys monism."⁴ (That is to say, it destroys a monism of substance, to replace it by a pluralism of substances, without, however, at all troubling to justify the substance category as ultimate.)

We next pass to what analysis reveals as a second type of proposition, in Mr. Russell's eyes, vastly more important than the preceding. These are what he calls "relational propositions" in distinction from the subject-predicate propositions, just considered. They are, in short, the form of proposition which, for him, constitutes mathematical science. In his criticisms of Leibniz, it will be remembered, Mr. Russell discusses one such proposition, namely, 'A is greater than B'. Here, according to our authority, we find by analysis that we have two terms, 'A' and 'B,' instead of the single subject-term of the ordinary type. In the language of mathematical logic these two terms are the 'constituents,' and 'greater than' is an expression of a relation between them. Now either the constituents, or the relation between them, may be changed independently, so the argument runs, without correspondingly altering the form—that of a two-termed relation—of the proposition. And similarly we may construct propositions of any number of terms and relations, the 'form' in each case depending on the particular number in question.

² *Prin. of Math.*, p. 448.

³ *Introd. to Math. Phil.*, p. 200.

⁴ *Prin. of Math.*, p. 44.

Terms and relations thus take on the character of independent existences, and the (assumed) fact of independent alterability in the one or the other factor Mr. Russell expresses in the famous doctrine of 'external relations'. That is, in short, "no relation ever modifies either of its terms. For if it holds between A and B, then it is between A and B that it holds, and to say that it modifies A and B is to say that it really holds between different terms, C and D."⁵ This means, of course, that both terms and relations are real in their own right, and that the latter are quite independent of the former and *vice versa*. As such they constitute the ultimate parts, whose sum, literally speaking, is the proposition. At this point Mr. Russell makes an interesting observation, of which, however, he fails to see the full implications. Namely, he says, "owing to the way in which the verb actually relates the terms of the proposition, every proposition has a unity which remains distinct from the sum of its constituents."⁶ The important logical problem presented by this observation, however, he dismisses as one which "deserves to be fully and thoroughly discussed."

Surely this is a most damaging admission! If, over and above the atomic parts, whose sum, according to our analysis, constitutes the proposition, there remains an acknowledged unity which escapes this analysis, then, obviously, just in so far the preceding account of the structure of the proposition is incomplete and the analysis a failure. Such admitted shortcomings and confessed failures, we are coming to learn, however, do not in the least dampen the ardor of the mathematical logician. Accordingly, we at once ignore and pass over these difficulties, as if they were of no real consequence, so far as the validity of the procedure in question is concerned, to the next step in that procedure.

The theory that the terms and relations of propositions may be varied at will, without thereby altering the logical significance of the proposition, is equivalent to saying that what matters about a proposition, from the point of view of mathematical logic, is its form and not its content. In other words, the results of analysis

⁵ *Prin. of Math.*, p. 448.

⁶ *Ibid.*, p. 52.

reveal the fact that "logic is concerned only with forms"; or, with what we may call, in view of their character of invariance in connection with any variable content, "logical constants."⁷ On this view the content, the meaning, of a proposition becomes a factor of no *logical* importance; indeed, it is in principle incidental and accidental, so far as the logician is concerned, whether there actually exist any 'empirical' content at all!

The principal forms or logical constants comprise the ultimate data of, and therefore may be studied conveniently in connection with, what constitutes for Mr. Russell the three fundamental subdivisions of mathematical logic, in its function as a general technique or method of inference, namely, the calculus of propositions, the calculus of classes, and the calculus of relations.

"A proposition," for the purposes of the calculus of propositions, "is anything that is true or that is false."⁸ 'Plato is a man,' is a true proposition, in this sense of the term, and 'Paris (the City) is a man,' a false one. But now suppose we substitute for the first term or constituent of the proposition a 'variable' such as X . Then the expression ' X is a man,' is no longer a proposition, since, while X remains undetermined, such an expression is neither true nor false. But it is "as it were a schematic form standing for any one of a whole class of propositions."⁹ For certain values of X we get a true proposition; for other values a false one. Such an expression, containing one or more variables, is called a "propositional function." For example, consider the universal proposition of ordinary formal logic, 'All S is P '. Interpreted in terms of propositional functions, S may be defined by the propositional function $F(x)$, and P by $G(x)$. Then 'All S is P ' means ' $F(x)$ implies $G(x)$ ' is always true. Similarly, 'Some S is P ' means ' $F(x)$ implies $G(x)$ ' is sometimes true.¹⁰

That is to say, Mr. Russell interprets the universal judgment as a statement regarding the several members of a class, taken in extension. The connection between the subject and predicate is therefore regarded as purely accidental; from the point of view of

⁷ *Introd. to Math. Phil.*, p. 199.

⁸ *Prin. of Math.*, p. 13.

⁹ *Loc. cit.*

¹⁰ *Introd. to Math. Phil.*, p. 161.

logic only the 'form' and not the content really matters. But it is hard to see, on this interpretation, what remains of the acknowledged unity of the proposition. What this unity should mean, it would seem from our study of Leibniz, is that there is a necessary relation between the content of the subject and that of the predicate; and that consequently form and content go hand in hand. And the only semblance of reality attaching to Mr. Russell's doctrine undoubtedly comes from the fact that, in mathematics—the domain of quantity—we have exhibited those judgments which most nearly approximate—without, however, ever attaining—to a purely extensional character.

Now, as we have seen, among the particular values attributed to the variable of a propositional function, some give rise to true propositions and others to false ones. The totality of values for which a propositional function becomes true determines a class. So that "a class (which is the same as a manifold or aggregate) is all the objects satisfying some propositional function."¹¹ And at this point we accordingly are ready to study the calculus of classes. It follows at once from the preceding that if two propositional functions are equivalent, the corresponding classes are equal (identical). Thus "equality of [the classes] a and b is defined by the equivalence of ' x is an a ,' and ' x is a b ,' for all values of x ."¹² Another important point is that we must not confuse a class comprised of only one member with that member itself. The fact is, that an individual, as an ultimate particular, directly perceptible, is therefore not definable. We may, however, define the identity of individuals. Two individuals are said to be so identical when the second belongs to every class of which the former is a member. This identity, we are told, must somehow be kept logically distinct from the equality of classes, though both are evidently considered in extension, *i.e.*, without regard to the content of the terms in question. But how this may be, though of some importance, is left unexplained, except for the following subterfuge. Namely, individuals, of course, exist, but classes, it appears, are only convenient shorthand symbols or "logical fic-

¹¹ *Principia Mathematica*, I, p. 24.

¹² *Prin. of Math.*, p. 21.

tions." This means that if any proposition in which a class occurs be analyzed correctly, we shall find that only particular terms remain. In a completely logical language, "with a definition for everything definable, and an undefined symbol for everything undefinable . . . all the particular things there are in the world would have to have names which would be included among undefined symbols,"¹³ while everything else would be defined in terms of these particulars.

So much for the self-evident particulars. But it might be thought that classes could be regarded as unitary wholes, and hence as real existences in that sense, but there are two objections to this view. First, if there were such wholes, then there would be one all-comprehensive class, Mr. Russell argues, which would embrace everything, and consequently also itself, as one of its members. That is, "if there is such a thing as 'everything,' then 'everything' is something and is a member of the class 'everything'." Most classes, however, are not members of themselves. The human race, for example, is not itself a member of the race of men. So we can form an assemblage of all classes not members of themselves. Is this new class a member of itself or not? "If it is it is one of those classes that are not members of themselves, *i.e.*, it is not a member of itself. If it is not, it is not one of those classes that are not members of themselves, *i.e.*, it is a member of itself." So each of these two hypotheses implies its contradictory and the whole conception of some one all-embracing whole is therefore self-contradictory.¹⁴ Secondly, if classes were real in the same sense as particulars, it could be shown that the number of classes would be greater than the number of individuals comprised in them, and hence could not themselves be individual wholes.¹⁵ Then again, the attempt has been made to consider classes as heaps or conglomerations—a mere plurality. But on such a basis it is impossible to account for the null-class, "which has no members at all," and which is yet the necessary 'logical' foundation for the mathematical zero. It is also very hard, we

¹³ *Introd. to Math. Phil.*, p. 182.

¹⁴ *Ibid.*, p. 136.

¹⁵ *Ibid.*, p. 183.

have seen, on such an extensional view, to understand how a class of only one member may be distinguishable from that member.¹⁶ And infinity only furnishes added difficulties.

The final result, then, of Mr. Russell's prolonged consideration of this problem of the nature of classes is that they are manipulative conveniences, which play their part in logical computations, but completely drop out when analysis is strictly carried through to the end. This leaves as real only the ultimate particulars, of which, as Descartes would have said, 'everything is composed'.

Let us consider for a moment the significance of this result. Historically it marks the passing of a notion which, from the time of Boole on, has been of considerable importance in the logical calculus. And it would seem that a calculus operated on the basis of the extension of terms would naturally have to presuppose this notion. Mr. Russell's frank exposition of the contradictions and limitations which he finds in the idea of an extensionally defined class therefore really pronounces judgment on the purely extensional viewpoint. Yet we know that mathematical logicians themselves confess that they can operate their calculus on no other basis. Mr. Russell indeed makes a gallant effort to save the day by introducing the complications involved in the use of logical fictions. But assuredly the only legitimate way to show that an idea really is a fiction is to proceed without it; and this, of course, he is quite unable to do.

Moreover, we may just note in passing the admission of the necessary factor of continuity in connection with this evidently quantitative concept of a class. The factor of continuity is introduced, implicitly, in the (imperfect) guise of a propositional function, which, we find, is "the only way of expressing a common property generally."¹⁷ Now the phrase 'common property' evidently represents a rough substitute for what Leibniz and other thinkers have understood by the logical factor of continuity. How far this admission is consistent with another statement (more in keeping with the main standpoint of mathematical logic), to the

¹⁶ *Loc. cit.* Cf. Brunschvicg, *Etapes*, pp. 394-411.

¹⁷ *Introd. to Math. Phil.*, pp. 157-158. In his *Logic*, II, pp. 66-75, Mr. Johnson convincingly points out many difficulties in this conception.

effect that all common properties can be analyzed into independent terms and relations,¹⁸ we leave it to Mr. Russell to explain.

But yet, in still another fashion, Mr. Russell is our authority for the contention that parts logically imply wholes, or that wholeness is the necessary complement to particularity. In his chapter on the "Theory of Logical Types,"¹⁹ the author points out that we must distinguish between a genuine totality and "a totality containing members defined in terms of itself," *i.e.*, as abstract counterparts of itself. The latter kind of totality is stigmatized as "illegitimate."²⁰ In our own language, following Leibniz, we should say that an illegitimate totality—such as mathematical infinity, or a 'logical class'—is a whole of mere quantity, which necessarily contradicts the very 'quality' of quantity. A 'genuine totality,' on the other hand, contains members defined, not in terms of themselves, but in terms of their participation in the whole, as its organic constituents, as diversity in unity. Why Mr. Russell himself did not learn this lesson from Leibniz, and why he does not now see that the "inchoateness" and "obscurity" which, he confesses, still permeates his doctrine of 'logical types,'²¹ is due to his own self-condemned theory of external relations and independent particulars, we fail to understand. And our failure is greatly magnified by his further admission that a complete analysis, such as is necessary to reveal the ultimate particulars—in another place viewed as given to immediate perceptual intuition—is only ideally possible; "that whatever appears as a particular is [possibly] . . . some kind of a complex."²² How much of the original structure of mathematical logic will ultimately remain after Mr. Russell's various re-considerations of its problems, it will be interesting to observe.

We come, finally, to what constitutes for this thinker the third and last of the general subdivisions of mathematical logic, namely the calculus of relations. This calculus is the newest and both

¹⁸ *Prin. of Math.*, p. 226; *Scientific Method in Phil.*, p. 42; etc., and below in this chapter.

¹⁹ *Principia Mathematica*, I, Ch. II.

²⁰ *Ibid.*, p. 67.

²¹ *Introd. to Math. Phil.*, p. 135.

²² *Ibid.*, p. 142.

from the point of view of mathematical logic, and of a Realistic metaphysics, the most important and general part of the subject. We have seen how, for Mr. Russell, the vice of the traditional logic was its adherence to the subject-predicate form of proposition, and we have also examined the theory of relational propositions, which, it was claimed, was able to effect a great generalization of this position. We have already criticized his interpretation of Leibniz and traditional logic generally, in this regard; we have next to study more in detail his own positive theory of the various types of relations, or 'logical constants,' which may occur in such relational propositions.

At this point, however, it may be well to mention Royce's attitude towards this doctrine of relations. Royce, we find, claims—perhaps influenced by a subconscious apprehension of the difficulties we have found to inhere in Mr. Russell's conceptions—that Mr. Russell's basic distinction between the calculus or logic of propositions, classes and relations, is, in the light of his theory, quite a superficial distinction.²³ "Give us a system"—i.e., the 'system Σ ' already mentioned—"of entities, of the types of logical classes," Royce explains, "and we shall find that their relations . . . [all statable in terms of a generalized 'between' relation] are already (quite apart from any separate 'logic of relations') certainly as rich as the totality of the relations studied in geometry," and indeed, "are probably as rich as the totality of the relations known to the exact sciences. . . ." By this he seems to mean that the 'entities' considered by any such science, together with the relations that exist between them, will be found by analysis already to be contained in the original 'logical' system; from which, indeed, they may be derived simply by a judicial 'selection'. By this simple manœuvre, therefore, Royce (apparently) does away with, or renders unnecessary, Mr. Russell's division of logic into the three calculuses of propositions, classes and relations.

One illustration of the force of this distinction is that whereas Mr. Russell will 'deduce' cardinal number from his logical concept of a class, Royce derives ordinal number from the logical con-

²³ "The Relation of Logic to the Foundations of Geometry," *Trans. of Amer. Math. Soc.*, vol. VI, pp. 353-415; cf. esp. p. 358.

cept of a relation. And hence the disagreement, of which mention already has been made (Chap. III), as to the relative priority of cardinal and ordinal numbers.

But however that may be, there is mutual agreement, among contemporary mathematical logicians, as to what constitutes the essential properties of the relations themselves. Important among such properties are symmetry, transitivity, and asymmetry and intransitivity. For example, in studying the mathematical concept of order we found that we might say that one term 'preceded' another. To the 'relation,' then, upon which such an order is said to depend, these thinkers assign the following properties: If X precedes (or is to the left of, or is earlier than, or is less than) Y, then the relation expressed by these words 'precedes,' etc., is said to be *asymmetrical*. Obviously, in such a case Y cannot also precede, etc., X. On the other hand, if X is of the same height as Y, Y is also of the same height as X, and such a relation between the terms X and Y may then be called *symmetrical*.²⁴ *Transitivity* is a second property which may be illustrated as follows: If X precedes Y and Y precedes Z, then X must precede Z. But if X is the father of Y and Y of Z, then obviously X is not the father of Z, and hence the relation expressed by the phrase 'father of' is *intransitive*.²⁵

All relations (be it noted!) must have *converses*: e.g., if X precedes Y, then Y follows X; or if X is the father of Y, then Y is the child of X. It follows that "a series is the same thing as a serial relation." Thus 1, 2, 3; 1, 3, 2; 2, 3, 1; 2, 1, 3; 3, 1, 2; 3, 2, 1; are six different series, dependent upon as many different (asymmetrical, transitive) ordering relations.²⁶

In other words, the ordinal aspect of any aggregate (in the form of a numerical series), and hence ordinal number itself, may be defined, according to our authorities, in terms of the properties of certain logical constants, i.e., relations. Especially important among the special types of such logical constants is what they technically call a 'between' relation. Such a relation holds between three terms. For instance, of three points on a

²⁴ *Introd. to Math. Phil.*, p. 31.

²⁵ *Loc. cit.*

²⁶ *Ibid.*, p. 34.

straight line, one point must be between the other two.²⁷ The same idea may be applied, with appropriate modifications, to the generation of numerical series. Thus it seems that, in the final analysis (literally speaking), "the minimum ordinal proposition, which can always be made, whenever there is any order at all, is of the form: 'Y is between X and Z'"; and this means that "there is some asymmetrical transitive relation which holds between X and Y and between Y and Z."²⁸ Obviously, then, it follows that geometry, viewed as a branch of pure mathematics, may be considered as "the study of series of two or more dimensions." And in this fashion, for Mr. Russell as for Royce, pure geometry, like arithmetic, is held to be deducible directly from logical premises, without any necessary appeal to spatial intuition.²⁹

But a clear understanding of how this may be presupposes a knowledge of the way in which the so-called logical properties of terms and relations are associated. We are now prepared to approach this question.

It is self-evident that a relation presupposes the particular members of the class or classes—in the case of Royce, the elements of the base-system—between which it holds. Nevertheless it appears that the properties of a relation (such as those just enumerated) may be studied quite apart from the nature or meaning of the elements or terms, and are independent of them. This is, in fact, as we have seen, the very meaning of the phrase 'external relations'. As for the properties of a term, they, according to Mr. Russell, are simply "other terms to which it stands in some relation; and a common property of two terms is a term to which both stand in the same relation."³⁰ Or, as he explains in another place, "where a group of objects have that kind of similarity which we are inclined to attribute to possession of a common quality, the principle in question [*i.e.*, the principle of abstraction] shows that membership of the group will serve all the purposes of the supposed common quality, and that therefore unless some common quality is actually known, the group or class of similar objects may be used to

²⁷ *Op. cit.*, pp. 38-41.

²⁸ *Prin. of Math.*, p. 217.

²⁹ *Ibid.*, p. 372.

³⁰ *Ibid.*, p. 226.

replace the common quality which need not be assumed to exist.”³¹ For example, we may think of two collections as possessed of a common property called their number. We say, then, that the number exists. But upon analysis we find that being equally numerous is really a certain kind of relation of the two collections, as is the case also with height or weight or color. That is to say, if two collections have the same number, this may be expressed logically by saying that a transitive symmetrical relation holds between the class comprised of the members of the one collection and that composed of the members of the other collection. In any case in which we were compelled to use the notion of number, therefore, we could replace it by a statement concerning the relation between the classes involved. This precludes the necessity of assuming the existence of any such entity as a number, and hence supports the good old maxim, ‘entities are not to be multiplied without necessity’.³² The conclusion is, therefore, that we may dispense with the idea of common properties entirely, since “the class of terms that have the [appropriate] . . . relation to a given term will fulfill all the formal requisites of a common property of all the members of the class.”³³

The advantage of this procedure is that it is supposed to obviate the necessity of solving the difficult problem as to whether certain common properties actually exist or not; as the name, ‘principle of abstraction,’ denotes, mathematical logicians profess to abstract from all reference to the qualitative aspect of terms, and consider only those terms which are wholly devoid of any sort of complexity, *i.e.*, of all content, or meaning.

Now all this is very important as well as very difficult. In general, the idea of dispensing with common properties rests upon the principle that the possible is wider than the actual. And, moreover, it presupposes as a basis the purely extensional reading of terms, which, we have seen more than once, is so characteristic of mathematical logic. And since classes are ‘logical fictions,’ such a concept as a number turns out to be, in the same sense, a

³¹ *Scientific Method in Phil.*, p. 42.

³² *Introd. to Math. Phil.*, pp. 18, 184. Cf., *e.g.*, below, Ch. VI, on the definition of cardinal number.

³³ *Scientific Method in Phil.*, pp. 125-6.

fiction of a fiction, *i.e.*, at two removes from reality in the form of ultimate particulars.³⁴

In addition to these complexities and difficulties, what becomes of the notion of a propositional function, which, we have learned, is "the only way of expressing a common property generally?"³⁵ That is to say, propositional functions determine the membership of classes; *i.e.*, a class is defined as all the terms satisfying some propositional function. Yet in so far as we dispense with common properties, it would seem that we automatically dispense with propositional functions also, and hence with the necessary foundation of one whole section of the logical calculus.

Or, in other words, in asserting that a class of terms having a given relation to a given term will fulfill all the requisites of a common property of the members of the class, Mr. Russell is guilty of arguing in a circle. This we shall presently find him denying, but that does not alter the plain fact, and under the circumstances his very solicitude to deny only re-enforces the argument.³⁶

So much for the properties of the terms. Now for the properties of relations which hold between terms. As Mr. Russell says, we are not only aware, by direct sense-perception, of particulars, say a spot of yellow; we are also intellectually aware of universals, say yellowness. And universal relations, too, are objects of the kind of awareness that is called conceiving or intellectual perception.³⁷ Think for example of such spatial relations as 'inside' and 'outside,' or 'before' and 'after'. One might suppose that we do not know directly, by intellectual perception, the properties of a relation, even though prepared to admit that we know particular terms directly through sense-perception. Thus, it is sometimes argued that we do not know the relation apart

³⁴ Mr. Russell, I believe, is himself authority for this inference, though I am unable to find the exact reference. At all events, it is an obvious one.

³⁵ Cf. above, in this chapter, the section on classes.

³⁶ So far as Royce is concerned, the fact that his original elements, or terms, are assumed, to begin with, to have no meaning whatsoever, obviates for him the necessity of a discussion of these problems—or rather, it naturally raises the question as to the ultimate validity of such a subterfuge.

³⁷ *Mysticism and Logic*, pp. 212–213.

from a complex (or whole) in which it occurs, as 'this is before that'. (The remainder of Mr. Russell's argument we repeat verbatim.) "This view," he proceeds, "... is difficult to reconcile with the fact that we often know propositions in which the relation is the subject, or in which the relata are not definite given objects, but 'anything'. For example, we know that if one thing is before another, and the other before a third, then the first is before the third; and here the things concerned are not definite things, but 'anything'. It is hard to see how we could know such a fact about 'before' unless we were acquainted with 'before' and not merely with particular cases of one given object being before another given object."³⁸ Since the properties *ex hypothesi* do not depend upon the terms—the latter in fact not necessarily possessing any definite properties at all—they must belong, so the argument runs, to the relation itself. And since we cannot perceive 'before' with our senses, as we can a sensible object, and yet admittedly know its properties, we must be capable of grasping these by intellectual perception. In other words, direct sense-perception, say of a flash of color, and the knowledge of the meaning of concepts—universals, including relations—say 'yellowness,' or 'before,' are two distinct and separate instances of what is called "self-evidence." And the evidence for self-evidence itself is just that, to Mr. Russell's mind, it is necessarily presupposed by the knowledge which we actually possess, both of particulars and of universals, of terms and relations. This, in particular, is the only way of rendering coherent our knowledge of mathematical propositions.³⁹

Spatial relations, for example, satisfy certain conditions, or possess certain properties, as exemplified in the special case of 'before'. This idea Mr. Russell believes to be "self-evident." "These conditions are not demonstrable by purely logical considerations: they are synthetic properties of [intellectually?] perceived relations."⁴⁰ That is, as we have seen, relations are among the ultimate indefinables or primitive ideas of mathematical logic.

³⁸ *Op. cit.*, p. 213.

³⁹ *Cf. Ibid.*, pp. 209-232; and other works.

⁴⁰ "On the Relation of Universals and Particulars," *Arist. Soc. Proceedings*, 1911-1912, p. 17.

Furthermore it appears that it is upon this basis of intellectually perceived spatial relations that the numerical diversity of two patches of color, say yellow, also is self-evident. "They have the relation of being outside each other, and this requires that they should be two, not one." Whether or not the patches have intrinsic qualitative differences is of no importance for mathematical logic; indeed, "it is obviously logically possible that they should have no intrinsic differences whatever. It follows from this that the terms of spatial relations cannot be universals . . . but must be particulars capable of being exactly alike and yet numerically diverse."⁴¹

Numerical diversity, as contrasted with qualitative differences, is thus taken to be the necessary and sufficient condition for difference in terms; it is, "as its name implies, the source of plurality."⁴² But conceptual diversity, or difference between universals—*e.g.*, relations—though it implies numerical diversity, is something more. "This may be characterized by the fact that two propositions in which the concepts occur otherwise than as terms, even if, in all other respects, the two propositions are identical, yet differ in virtue of the fact that the concepts which occur in them are conceptually diverse." Such is the difference between the two propositions 'A is greater than B' and 'A is earlier than B'. Though conceptual diversity is less important to mathematics than numerical diversity, the former is "fundamental in logic."⁴³

We may illustrate this notion of conceptual diversity by another example. So far we have treated only of symmetrical transitive relations. In so far as they are concerned, a common quality of the terms, when it is known, may be used as a predicate to take the place of the relational expression. Suppose we say, 'This is red,' 'That is red,' etc., etc. These are 'subject-predicate' propositions (*i.e.*, propositions in which we assert an attribute of a single term, the subject), but they may be reduced to symmetrical transitive relations by means of the common quality of color likeness,

⁴¹ *Loc. cit.*

⁴² *Prin. of Math.*, p. 47.

⁴³ *Ibid.*, pp. 46-7.

i.e., if 'This' is of the same color as 'That,' then the converse holds, and if 'This' and 'That' are each of the same color as some third standard, then they are of the same color with each other. So that it means the same thing, logically, to employ either the subject-predicate propositions or symmetrical transitive relations; though for the sake of uniformity of procedure, if for no other reason, it is preferable even here to adopt the relational view.

But not all relations are of this type. Notably, there are also asymmetrical relations of which 'A is greater than B' is an example. Here we have two terms, A and B, which—always according to our authorities—do not have a common predicate or property. Neither can we say simply that they have different predicates. For if A has a different predicate from B, the converse is equally true, and either sameness or difference of predicates is a symmetrical relation, whereas what we have here is an asymmetrical relation. That is, we cannot say, as we can in the case of symmetrical relations, both that 'A is greater than B' and that 'B is greater than A'. "This impossibility of reducing dyadic relations [relations between two terms] to sameness or difference of predicates is a matter of a good deal of importance in connection with traditional philosophy because a good deal of traditional philosophy depends upon the assumption that every proposition is of the subject-predicate form, and that is certainly not the case."⁴⁴ And as we have seen already, in part, "asymmetrical relations are involved in all series, in space and time, greater and less, whole and part, and many others of the most important characteristics of the actual world. All these aspects, therefore, the logic which reduces everything to subjects and predicates is compelled to condemn as error and mere appearance."⁴⁵ Acknowledge the reality of asymmetrical relations, on the other hand, and only prejudice and respect for outworn tradition will prevent our admitting the full reality of the world of sense-perception. Asymmetrical relations then, as Mr. Russell says, are "the most characteristically relational

⁴⁴ "The Philosophy of Logical Atomism," *Monist*, Vol. XXIX, pp. 36-7.

⁴⁵ *Scientific Method in Phil.*, p. 49.

of relations, and the most important to the philosopher who wishes to study the ultimate logical nature of relations.”⁴⁶

Let us follow this argument on the properties of relations through again in the same order in which we have set the various points down.

Take first the point establishing knowledge of the properties of relations by direct acquaintance. As our authority himself says, this is an extra-logical source of knowledge, and hence unavoidably subjective and psychological. It is therefore at the mercy of any rational argument which can display its falsity. Now Mr. Russell holds that certain properties belong to relations quite apart from their terms—the logical converse of the view that terms may not have any properties at all. Yet in the case of such a ‘relation’ as ‘before,’ or ‘precedes,’ for example, there must be terms between which, as he likes to say, the relation holds. Even if it were granted that these terms need not be ‘definite,’ in Mr. Russell’s sense of the word, this would be no denial in principle of their inevitable presence if we are to distinguish relations at all. So much, at least, no one has denied. But surely with this admission the whole argument about the unqualified independence of relations falls to the ground, and with it the notion that we are acquainted with such relations by a pure intellectual intuition. For it is argued that at least some terms, such as a spot of yellow, are apprehended by sense-perception. And if so, then the distinction between logical and extra-logical, or merely psychological knowledge, likewise ultimately breaks down, at least in the form in which Mr. Russell maintains it.

The second point is curious. The terms of spatial relations, it is argued, are particulars capable of being exactly alike, and yet numerically diverse. Now Hegel is often accused of being a paradoxical writer, but one would search far in his works for a paradox to equal this one. Numerical diversity does not belong to the relations—so Mr. Russell insists—it is a property of the terms—so he is careful to emphasize. Yet in the same breath he also insists that the terms of relations need differ in no respect whatever. Hence we are confounded with the paradox of a diversity

⁴⁶ *Introd. to Math. Phil.*, p. 45.

which is no diversity, or of an exact identity which yet manifests differences. And this paradox, we willingly admit, is extra-logical—that is, on Mr. Russell's principles, it has no logical solution.

The clue to the solution of these problems is, however, not far to seek, and becomes obvious with the next step in the argument. Namely, it lies in the false distinction which our authority makes between conceptual diversity and numerical diversity; between universals and particulars; in other words, in the complete separation between relations and terms. He admits that propositions of the 'form' of symmetrical transitive relations, may be reduced to propositions of the subject-predicate form, or *vice versa*, but contends at the same time that such is not the case with regard to asymmetrical transitive relations. For our part we accept neither the admission nor the contention. That is to say, Mr. Russell's conception of the meaning of the so-called subject-predicate form of proposition has already been shown to depend on a false principle—that of a substance-accident view of things. Consider for example his illustrative proposition, 'This is red'. 'This,' he says, is a particular, pure and simple, while 'red' is a universal, also pure and simple. Now such a view is, in effect, pure and simple nominalism. It contends that a unique, particular 'This' is distinguishable from all other particulars, and yet that in principle it possesses no properties of its own. Here is another paradox, equally insoluble with the preceding. But both paradoxes dissolve if we refuse to accept the conception of unique, yet qualityless particulars, and hold rather that particularity as such is but one factor, whose necessary complement is universality. 'This,' to be distinguishable, must be distinguishable from something, so that we may discern it as 'This is not that'; but such distinguishability also presupposes something in common, an element of universality, between 'This' and 'that,' or whatever it is distinguished from. Numerical diversity permits—according to Mr. Russell—in addition to such diversity, a common quality. We say rather, not simply 'permits' but 'presupposes'; and *vice versa*, we grant that numerability as such presupposes separate objects capable of being enumerated. In short, this contention about the inseparability of the aspects of universality and particularity is but the

broader view of the inseparability of the factors of discreteness and continuity inherent in all quantitative determinations. In other words it is not a question of conceptual *versus* numerical diversity, but of qualitative *versus* quantitative diversity.

And with this we go on to consider next the distinction between symmetrical transitive relations and asymmetrical transitive relations. The former form, it is asserted, can be reduced to propositions of the 'subject-predicate form,' while the latter cannot. But now suppose we set down in order the three propositional forms together; *e.g.*, 'This is red (color),' 'A is equal to B,' and 'C is larger than D'. The first 'form' may be transformed (we have learned) into the second, as follows: 'This is of the same (red) color as that'; and *vice versa*, the second form may be reduced to the first thus: 'A is three inches long' and 'B is three inches long'. So far Mr. Russell. But then, on the very same principle, what prevents the transformation of the third proposition into, say, these two: 'C is six inches long,' and 'D is five inches long'? In each case we have a 'common property,' either a color or a spatial dimension, on the basis of which we effect a comparison between two or more objects. So that the distinction between asymmetrical and symmetrical relations so far breaks down. Both types exemplify the same 'form' of proposition, and in no wise necessitate a revision of traditional logic, so-called. Or, more precisely, as Leibniz taught, the real subject of such comparative propositions as 'A is greater than B' is neither 'A' nor 'B,' but the universal which each exemplifies; while the complete subject of such a proposition as 'This is red' is not a blank 'This,' with no qualifications whatsoever, but, again, 'This,' with a universal as well as a particular aspect. So that the assertion that an asymmetrical relation always implies its converse should be completed by another assertion to the same effect with regard to symmetrical relations, for it is as obviously true and necessary in the one case as in the other. Equality, for example, is as completely a ratio as is inequality. In fact, it is all a matter of comparison or relativity, of objects or other phenomena with regard to one another, in some relevant whole or context. The distinction between symmetry and asymmetry, in other words, is wholly a verbal one.

As a conclusion to this chapter it will be very fitting, as well as very instructive, to consider briefly these results of Mr. Russell's 'analysis' in the light of our study of Leibniz's account of the nature of quantitative determination. To begin with, the very conception of externality itself is obviously nothing more nor less than a spatial metaphor, which it is ludicrous to think of applying, uncritically and unqualifiedly to the relation, say, of mind to its objects. It is easy to see psychologically, however, that an enthusiastic, speculatively inclined mathematician, if not too critical, might come to formulate a complete metaphysical theory based on the attributes of the domain of quantity. Not only would there be the historical warrant of Pythagoreanism to refer to; there is also the example of thinkers in the other natural sciences, notably physics and biology, to stir the unconscious spirit of emulation, residing, we may presume, even in the breasts of mathematicians. Arguing by analogy, we should expect an 'energetics' or an 'evolutionism' to be paralleled, sooner or later, by an 'arithmetization' of reality. In all such cases, obviously, we meet with the same psychological phenomenon: namely, the tendency to hypostatize favorite conceptions or categories. And granted the validity of this psychological analysis, what would be more natural, more inevitable, than the formulation of principles and conceptions like logical atomism and the externality of relations?

For the essence (or 'quality') of quantitative determination, according to Leibniz, resides in the mutually exclusive character of self-repellent elements. Qualitative differences between these elements being ignored, or almost entirely ignored, the most significant grounds of determination are lodged, instead, in some extraneous consideration, transcending, or 'external' to, pure quantity as such. The ultimate significance of the quality of endlessness characteristic of mathematical infinity, it will be remembered, has its explanation in this fact. Infinity, it may be said, is the frank acknowledgment of the impossibility of constructing a genuine, self-maintaining whole, in terms of quantity. And once given these attributes of the domain of quantity, it is easy to see how the mathematical logician, if sufficiently uncritical of the real basis of his speculations, might attempt to construct a system of meta-

physical and logical conceptions such as those we have been studying. The only ultimate philosophical significance of such a construction, however, is to be found in the fact that it plainly furnishes an incidental and unintentional confirmation of the essential correctness of the Leibnizian account of quantitative determination, rather than in its dogmatically professed character as a reasonable account of the nature of things. Hence the question of its metaphysical and logical validity, in the proper sense of the words, is so far an open one, and one which still stands in need of an as yet unfurnished rational confirmation.

CHAPTER VI.

A DEDUCTION AND A SELECTION: THE PROCESS AND RESULTS OF LINEAR INFERENCE.

Conclusion.

The process of inference, as we have seen, may be differentiated into three moments, namely, the starting-point, the process proper, and the result. But what distinguishes inference as 'linear,' we discovered in our Introduction, is a peculiar interpretation of the nature and interrelation of these three moments.

In empirical science the starting-point is always provisional, open to modification, revision, or even rejection, to make room for some new consideration arising out of the actual growth of knowledge. The process proper is less easy to characterize, shortly or definitely. In fact it is a complex, consisting of an adaptation of any or all partial methods of reasoning—deduction, induction, the use of analogy, etc., etc.—in so far as these methods are found applicable to the particular subject-matter and problem in hand. In this process the starting-point undergoes explication, modification, and development, and the result appears, in a variety of senses, as an addition to knowledge. The conceptions, for example, upon which the theory of aggregates depends, represent such a provisional starting-point for one division of mathematical science. Upon the basis of this theory, as we have seen, it is possible to construct a mathematical edifice of which the new conceptions of the continuum and of infinity form a part. Thus certain old problems find a hitherto unknown solution, new problems arise; and, doubtless, in the end, this progress will even make necessary a new foundation.

But the advocates of the linear inference theory of reasoning are not content with such a state of affairs. What they demand—*cf.* once more our Introduction—is, in general, a fixed, inviolable starting-point, the eternal and immutable truths of early rationalism. Their ideal—though, as we saw in the preceding chapter, in detail, an ideal admittedly impossible of becoming

actual—is the establishment of an ultimate set of conceptions, superior to all change, and anterior even to the most elementary data of empirical science. The process of reasoning—demonstration, deduction—then becomes a mechanical manipulation of set, fixed rules and ideas, while obviously the result of the process can only be a reformulation of the premises, a monotonous reiteration of the original conceptions. Such, at least, has been the only achievement of representative examples of this form of reasoning in the past. And consequently, as the final stage of our present study, we have to discover whether or not the program of contemporary logisticians differs in these respects from those of their predecessors.

First as to the starting-point. For Mr. Russell, of course, this consists of the set of 'logical constants,' or primitive ideas—the indefinables—which result from his so-called analysis of the ordinary mathematical conceptions. In other words, these fundamental 'logical' notions are assumed to be anterior to the purely mathematical notions which form the basis of that science as ordinarily understood. And the fact that the primitive ideas or logical constants which represent the last elements of analysis are undefined, that their meaning is not fixed or determined, is taken to mean that they in no way affect the necessity of the deductive enchainment of propositions. They should be considered as pure symbols, contentless counters or marks. For example, we do not need to know the meaning of such a word as 'incompatibility,' which serves, we learn, as the basis of the propositional calculus. Enough that the other ideas of the calculus may be defined in terms of it.

In the words of Professor Lewis, such a state of affairs points to a "mathematics without meaning," true, as Mr. Russell picturesquely puts it, "in all possible worlds, not only in this higgledy piggedly job-lot of a world in which chance has imprisoned us."¹ In this sense of the word, evidently, "all definitions are purely nominal,"² and theoretically superfluous. But practically, this substitution of defined equivalents is pragmatically convenient,

¹ *Introd. to Math. Phil.*, p. 192.

² *Prin. of Math.*, p. 112.

since it prevents unmanageable complexity of the terms as the subject develops. That is to say, by such means a simple term—the *definiens*—is substituted for a complex term—the *definiendum*.³ For example, instead of repeating, each time we had occasion to use it in the calculus of propositions, the complex idea of the incompatibility-of-a-proposition-with-itself, we may substitute the simple idea of negation, arbitrarily defined as the (identical) equivalent of the complex idea.

Now in order to undertake a deduction, there are required, in addition to these undefined ideas and their defined equivalents, certain 'primitive propositions,' taken as indemonstrable or self-evident, in the sense already explained. These propositions serve both as premises from which the conclusions are drawn, and also as rules for conducting the process of inference.⁴ That is to say, just as the simplifying operation of the substitution of defined equivalents may be performed in the case of the primitive ideas, so here, with regard to the process of deduction, one proposition may be substituted for another proposition, or for complexes of propositions. The former type of substitution depends upon the fact that the original ideas are strictly meaningless. The nature of the primitive propositions upon which the second type of substitution depends, Mr. Russell informs us, is 'tautology'. This word he confesses himself unable to define, however, though he feels "thoroughly familiar with the characteristic of which a definition is wanted."⁵ And in *Principia Mathematica* the assertion is made that "it is through the law of tautology and its consequences that the algebra of logic is chiefly distinguished from ordinary algebra."⁶

Against such a combination of meaningless symbols, nominal definitions, and tautological propositions, we are naturally inclined to repeat the charge of Descartes, against the demonstrations of the later Scholastics, that "they only repeat the same thing without giving any real information," and the warning of

³ *Principia Mathematica*, I, p. 11.

⁴ *Introd. to Math. Phil.*, p. 150.

⁵ *Ibid.*, p. 205.

⁶ *Ibid.*, p. 14.

Leibniz, that "we cannot form ideas as we like, though it seems that the Cartesians are of this opinion."

So far the procedure of the mathematical logicians seems but to repeat, in new language, the erroneous ways of their predecessors. The next statement only adds a further extension to the analogy. For, because of their double functioning, Mr. Russell argues, the rules of deductive procedure, including the theory of deduction itself, "are of the very texture of mathematical reasoning; without them, or something like them, we should not merely obtain the same results, but we should not obtain any results at all. . . . They must be absolutely true, or else what we deduce according to them does not even follow from the premises."⁷ Or, as Royce would say, the very denial of certain assertions involves their re-affirmation. But why we need to 'deduce' anything from principles so wholly sufficient unto themselves as to be absolutely true, or to be involved in their own denial, quite passes our comprehension. That is to say, if these premises already show forth the absolute truth, how can anything more come from them, or be required of them? To require something more implies a lack of self-sufficiency as they stand which is very damaging to their character as absolute, while if anything more can come, it too must be absolute in its own right. In short, there seems to be no real function for inference to perform, since all the significant results appear to be already included, or given, in the premises. The only process involved is a manipulation of the premises, and the only possible result of such manipulation would seem to be a repetition of the primitive ideas and propositions.

However that may be, we are now ready to study at first hand the nature of the process of deduction, and its results.

Its nature may be described in a word. As Professor Lewis has it, "the main thing to be noted about this operation of inference is that it is not so much a piece of reasoning as a mechanical, or strictly mathematical operation for which a rule has been given. No 'mental' operation is involved except that required to recognize a previous proposition followed by the main implication sign,

⁷ *Op. cit.*, p. 191.

and to set off what follows that sign as a new assertion.”⁸ In other words, we encounter again the ideal of a mechanically operated game, so fatally fascinating to early rationalism and empiricism.

The results of this ‘game’ we may illustrate, with Mr. Russell’s assistance, by an example, taken from the calculus of propositions.⁹ The five primitive and ‘tautological’ propositions of *Principia Mathematica*, which are to serve both as premises, and as rules, of deductive inference, are as follows (letters always standing for propositions): (1) ‘ p or p ’ implies p —i.e., if either p is true or p is true, then p is true. (2) q implies ‘ p or q ’—i.e., the disjunction ‘ p or q ’ is true when one of its alternatives is true. (3) ‘ p or q ’ implies ‘ q or p ’. (4) If either p is true or ‘ q or r ’ is true, then either q is true or ‘ p or r ’ is true. (5) If q implies r , then ‘ p or q ’ implies ‘ p or r ’.

Now we are ready to proceed with the illustration. Suppose we wish to prove (demonstrate, deduce), on the basis of these five rules, that ‘if p implies q , then if q implies r it follows that p implies r ’.

Take the fifth rule, assuming that ‘not- p or q ’ is, by arbitrary definition, the same as ‘ p implies q ’. This rule then reads: (a) ‘if q implies r ’, then ‘ p implies q ’ implies ‘ p implies r ’.

But the fourth rule, when we substitute ‘not- p ’, ‘not- q ’, respectively, for ‘ p ’ and ‘ q ’, becomes: (b) ‘if p implies that q implies r , then q implies that p implies r ’.

Substituting ‘ P_2 ’ for ‘ p ’, ‘ P_1 ’ for ‘ q ’, and ‘ P_3 ’ for ‘ r ’, in the last proposition (b), it then becomes: (c) ‘if P_2 implies that P_1 implies P_3 , then P_1 implies that P_2 implies P_3 ’.

Now substitute ‘ P_1 ’ for ‘ p implies q ’, ‘ P_2 ’ for ‘ q implies r ’, and ‘ P_3 ’ for ‘ p implies r ’ in the fifth rule. It then reads: ‘ P_2 implies that P_1 implies P_3 ’.

Therefore, by (c), ‘ P_1 implies that P_2 implies P_3 ’; or, ‘if p implies q , then if q implies r , p implies r ’.—Q. E. D.

True to the principles presupposed, each step in this process

⁸ *Survey of Symbolic Logic*, pp. 284–285. Cf. also Shearman, *Symbolic Logic*, p. 31.

⁹ *Introd. to Math. Phil.*, pp. 150–151.

from the premise to the conclusion simply echoes the preceding step; there is nothing in the one which is not also in the other; it is a simple substitution of defined equivalents—at best a form of immediate inference, in which, *ex hypothesi*, nothing new is achieved. As such it may be picturesque, but it certainly has no interest from any other point of view. Indeed, the very notion that only a mechanical operation of ‘implication’ is involved, presupposes the invariance, unchanged, from the mechanism, of the constituents originally presented to it. Of mediate inference or reasoning in the true sense of the word, *i.e.*, a mediation, of two or more attributes by other attributes, establishing a rational connection where none was directly observable at first, there is here, *ex hypothesi*, no trace.

So much as to the general results, confidently to be hoped for, from such an automatic exchange of meaningless counters. But this part of our exposition would not be complete unless we offered, as a particular case, an example of a characteristic deduction in *Principia Mathematica* to illustrate concretely the way in which a specific mathematical notion may be ‘deduced’ from purely ‘logical’ ones. For this purpose we select the derivation of cardinal number.¹⁰

Cardinal number is defined, provisionally, as the domain of the relation indicated by the symbol ‘Nc’. A domain of a relation is the class whose members have this relation to something or other. For example, if the relation be expressed by ‘precedes,’ then the domain of the relation will be the class of all those things that precede something or other (100.02; 33.01; 30.01).

‘Nc’ is the relation of the class of referents of ‘sm’ to ‘sm’ itself (100.01). ‘Sm’ is the relation of classes which are cardinally similar.

In the terms of Mr. Russell’s vocabulary, two classes are cardinally similar when there exists between them a “one-one relation,” *i.e.*, “when if X has the relation in question to Y, no other term X’ has the same relation to Y, and X does not have the same re-

¹⁰ The numbers are those attached to the propositions in *Principia*. We follow the translation of the symbolism given in Professor Lewis’s *Survey*, pp. 346–349.

lation to any term Y' other than Y " (71.03). And the same authority remarks that "it should be observed that the number '1' is not used in these definitions."

The final definition of cardinal number, then, is as follows: The cardinal number of a class is the class of all those classes that are similar—*i.e.*, stand in a one-one relation—to it. And similarly a particular number "is anything which is the number of some class." For example, the number '2' is the number of the class of all couples.

Such definitions are not circular, Mr. Russell holds, since "we define 'the number of a given class' without using the notion of number in general; therefore we may define number in general in terms of 'the number of a given class' without committing any logical error."¹¹

Here, then, we have before us a 'deduction,' from the 'logical' notions of 'class' and a particular type of 'relation,' of the mathematical concept of cardinal number. It will be illuminating to compare the definition of cardinal number so obtained with the ordinary mathematical definition given in chapter III, above. There, it will be recalled, cardinal number was defined, in properly mathematical terms, as a characteristic of equivalent aggregates. That is, when the elements of one aggregate can be placed in reciprocal one-one correspondence with the elements of another aggregate, they are said to have the same cardinal number. Upon comparing this definition with the preceding 'logical' one we are struck by a certain degree of similarity between them. In particular, there seem to be much the same ideas involved in what our authority calls a 'logical one-one relation,' and what the mathematicians choose to call a 'reciprocal one-one correspondence'. True, Mr. Russell attempts to maintain a distinction between the 'terms' of the 'logical' concept of a class and quantitative elements such as those of an aggregate. The former, he urges, are not numerical at all. And it should be noted that this is a special application of a general rule, established in the preceding chapter, that we must differentiate between an individual as such and the class of one member which it composes.

¹¹ *Introd. to Math. Phil.*, pp. 15-19. Cf. also Lewis, *op. cit.*, p. 349. This, of course, is a good example of a definition by 'extension'.

With regard to such a distinction either one or the other of the following alternatives is true. Either the contention that there is no quantitative idea presupposed in the definition is just, in which case the definition is logically erroneous, for cardinal number itself is admittedly a quantitative concept; or, the contention is unjustifiable, in which case it is hard to see how this definition in logical terms differs from the ordinary mathematical one. But either horn of this dilemma is fatal to the logicist's position. In fact, it is perfectly obvious, to any candid observer, that this pretended deduction of a mathematical notion from purely logical ideas is simply a travesty on the correct mathematical establishment of the very same notion. By his Quixotic efforts to accomplish the obviously impossible the mathematical logician has only succeeded in proving, against his will, the old maxim, *ex nihilo nihil fit*. In other words, this latest form of the logicist's magic art is quite true to its permanent character, in that it presupposes in its starting-point the very results which it pretends to draw therefrom by a mechanical re-iteration of tautological expressions. The whole process is an excellent illustration of the elementary logical fallacy of arguing in a circle.

Will Royce's process of selection prove less disappointing than Mr. Russell's phantom deduction?

The starting-point of Royce's procedure, we remember, consisted simply in a set of postulates, whose function it was to establish, once and for all, the existence of certain 'logical entities' and certain 'relations' between them—a bottomless source of 'types of order'.

The six principles or postulates to which the system thus constituted—the 'system Σ '—must conform, are designed to assure the presence of these relations, and especially of the 'between' relation which—as the 'logical' basis of the concept of order—is, in the eyes both of Mr. Russell and Royce, of such supreme importance in mathematical science.¹²

¹² The postulates are substantially as follows: (1) If A is a collection of elements of a particular sort, called an O-collection, belonging to the base-system, then AB is such a collection, whatever collection B may be. (2) If, whatever element (b) of the collection B be considered, its connection with any collection D makes of the combination an O-collection, then so also is D.

The postulation of this base-system represents, therefore, the solution of what was for Royce the "central problem of the Theory of Order," namely, the determination—more correctly, the *definition*—of "the necessary 'logical entities,'" and "their necessary laws."¹³ For Royce, then, the fundamental task of 'logic' is ended when he has achieved so much. It is a comparatively easy matter, on this view, to 'select' from the all-inclusive, initially ordered system, by an arbitrary choice of assumptions which determine a special class of entities (*cf.* the many 'logics' of Mr. Russell's theory), the particular type of order exemplified by this or that branch of mathematical reasoning. More, it is to be anticipated, according to Professor Lewis, that study of this original system probably will reveal "new continents of order whose existence we do not even suspect." That is, since the terms or elements *ex hypothesi* have no definite meaning of their own, they may be "allowed to denote ambiguously *anything* which has the relations in question, or relations of precisely that type."¹⁴ So that, for Professor Lewis, mathematics becomes simply a universal method, applicable to all possible worlds, and capable in a sense of creating or discovering worlds of its own. Freed from all taint of definite meaning and special content, a "mathematics without meaning," an empty schema of pure form, which appeals neither to intuition nor to empirical data, nor even to conceptual meaning, is the highest goal to which this "Science of Order" aspires.¹⁵

Now the next step in the procedure—what corresponds to a 'deduction' for Mr. Russell—is as follows: If we subject the

(3) There exists at least one element in the base-system. (4) If an element (x) of the system exists, then (y) exists, such that it is different from (x). (5) Whatever pair (p, q) exists such that they are not identical, (r) exists such that while neither (rp) nor (rq) make up an O-collection (rqp) does. (6) If the element (w) exists such that any collection T combined with it is an O-collection, then an element (v) also exists such that $T(v)$ is an O-collection, and such that whatever element (t) be considered (vwt) is an O-collection. In sum, then, the 'O-collection,' or base-system, serves, primarily, the important purpose of defining the 'between' relation. *Cf. op. cit.*, pp. 366–367.

¹³ *Encyclop. of Philos.*, p. 125.

¹⁴ *Survey*, pp. 369–370.

¹⁵ *Ibid.*, pp. 359 ff.

elements of the base-system to a certain "*entirely arbitrary restriction (foreign to the first principles of our system)*", but quite capable of being satisfied by a due selection of its elements," then we shall, as a result, be able to define a line as a one-dimensional continuum.¹⁶

In his article on "Types of Order and the System Σ ," in the *Philosophical Review*,¹⁷ Professor Lewis describes precisely how a line may be 'selected' from the system. First he explains that the fifth principle of the system may be restated thus: "For every pair of distinct elements, there exists an element, distinct from both, between them." This is expressed in the same terms as Mr. Russell's 'between' relation. Arbitrarily call the elements 'points' and we have an ordered collection which is both continuous and dense, beyond which only one step further need be taken to put us in complete possession of a line. This step is to be taken by means of a 'selection'. Thus in order to be a line the collection must be not only continuous and dense; every fundamental segment of the collection must have a limit. Now "consider two selections from the collection, K and L, such that if k is any element of K, every element j such that $j <_0 k$ [*i.e.*, j precedes k with respect to the origin o] belongs to K, and every element l , such that for every element k of K, $l <_0 k$ is false, belongs to L. There is, then, an element, call it s , such that, for every element k in K, $k <_0 s$, and if l is any element such that, for every element k of K, $k <_0 l$, then $s <_0 l$. Such an element s is the 'sum' or 'upper limit' of K. . . . Hence every fundamental segment has a limit. Any collection thus characterized by a transitive unsymmetrical relation and continuous order deserves to be called a 'line'. Every pair of distinct elements in Σ determines such a line."¹⁸

We are prepared, of course, to see in this 'logical' characterization of a line precisely those features which distinguished the mathematical characterization.¹⁹ But it is difficult to conceal an

¹⁶ *Trans. Amer. Math. Soc.*, Vol. VI, p. 408. Italics mine.

¹⁷ Vol. XXV, pp. 407-419, esp. pp. 414-5.

¹⁸ This account is based on Royce's article in the *Amer. Math. Soc. Transactions*, above referred to, pp. 384-412.

¹⁹ Above, Ch. III.

impression that there is an air of *hocus-pocus* about the procedure by which, it is claimed, this result has been obtained.

We note, but without understanding the basis for it, that the originally postulated elements must be submitted to an "entirely arbitrary restriction," which is wholly "foreign" to the first principles or postulates of the base-system. Indeed, the very virtue of the procedure seems to reside in the arbitrary or extra-logical nature of the restrictions which may be imposed upon it from without, and then artificially coupled with a "due selection of the elements" of the original system.

Just as there is an appeal by Mr. Russell to an extra-logical source of knowledge—*i.e.*, a direct intuition—so here we have an appeal to the extra-logical ground of arbitrary choice. And these appeals, though couched in different language, as a matter of fact are directed to precisely the same court, namely the court of developing mathematical experience and conceptions. Mr. Russell can 'deduce' cardinal number from the 'logical' notion of a 'class' only because the mathematicians employ the conception of an aggregate, etc., for that purpose. And Royce can 'select' a line from his abstract base-system only because the mathematicians have determined independently what the properties of a line shall be in terms of their present-day conceptions. In each case the logisticians have surreptitiously introduced into their presumably more ultimate conceptions and data, as presuppositions, the very conceptions which the mathematicians have previously taken pains to establish for themselves.

But mathematics is not a dead science; its fundamental conceptions and assumptions are subject to all sorts of transformations. We may anticipate, therefore, that at some future date numbers and lines will be subjected to new determinations, in some respects quite different from their present ones. In that case what will become of the *ex hypothesi* definitively fixed, because purely 'ideal,' 'a priori' conceptions of the mathematical logicians? To revise them in accordance with the developments in pure mathematics will be to admit defeat, so far as their claims of absolute certainty and of the independent, *a priori* character of their ideal starting-point is concerned. Yet not to revise them will be to

ignore the progress of the very science whose foundations they claim to have secured. In other words, it is now apparent that Mr. Russell's resort to direct 'intuition' and Royce's dependence on the arbitrary choice of an external 'selection' is a tacit admission of the fact that both thinkers are dealing with hypostatized conceptions and unreal, verbal abstractions. A "mathematics without meaning" actually is proving itself to be without meaning for mathematics.

And, frankly enough, Mr. Russell for his part, and in principle, more than once admits as much. This fact M. Brunschvicg calls to our attention in his *Etapas de la philosophie mathématique*. There he quotes from his original authority both direct and indirect confirmation of this assertion. Thus Mr. Russell says that "several of the ultimate premises have less intrinsic evidence than many of the consequences which are deduced from them."²⁰

This, however, may be said to be only indirect evidence. But there is still more. For the same authority complains that there has been "a misunderstanding as to the pretensions of Logistic and the nature of the evidence on which it rests. . . . The method of Logistic is essentially the same as that of all other sciences. It lays claim to the same fallibility, and the same incertitude, the same necessity of appealing, in confirmation of its principles, to the general agreement of calculated results with results of observation."²¹ In short, it is, or should be, the objectively grounded intuition of the instructed mind, and not the intuition of individual experience as such, which guarantees the validity of the first principles of the science.²²

On the vast differences between the standpoint implied by these remarks, and the standpoint actually adopted in the preceding pages, it is not necessary to dwell. These latter remarks require little modification in order to give expression to the views to which we ourselves would subscribe as outlining a secure basis of regular scientific inference. But unfortunately the actual doctrines and claims both of Mr. Russell and of Josiah Royce are manifestly

²⁰ *Op. cit.*, p. 422. And cf. above, Ch. V, where we quote an assertion to the effect that the simple propositions of mathematics have as much self-evidence as those of pure 'logic' itself.

²¹ *Ibid.*, p. 411.

²² *Principia Mathematica*, I, p. 13.

quite contrary to such a point of view. The fruitless endeavor to prescribe a basis of inference anterior to scientific experience, as such, can make no valid appeal, in justification of its inordinate claims, to the results admittedly based, directly or indirectly, upon the relatively given data of experience, and legitimately obtained by the usual experimental methods of mathematical science.

A review, inclusive alike of the historical background and of the most ambitious modern forms of mathematical logic, is hardly more than a record of a complete failure.

In the first place, the modern logisticians founded their theory on a misinterpretation or misunderstanding of the import of Leibniz's philosophy—a misunderstanding which spread until it included various important aspects of philosophy at large. This was accompanied by a corresponding failure to comprehend the principles, the methods, and the fundamental concepts of mathematical science. And from these roots there sprang a most unhappy *mésalliance* between the sciences of mathematics and logic themselves. A logic or epistemology which may by hypothesis undertake no untrammelled examination of the nature of thought and its products, and a mathematics artificially harnessed to a set of hypostatized, verbal fictions, are the only fruits of this standpoint. Finally, the logisticians not only missed the significance of the so-called subject-predicate form of proposition; they also have failed to grasp the empirical significance of quantitative determination; while perhaps the most fatal error of all, because the most extensive in its effects, is their entire lack of insight into the doctrines of speculative philosophy as to the nature of mind and its relation to experience.

These negative errors, moreover, are balanced by a similar failure to achieve any positive results. The method of linear inference, or progressive deduction, we have shown in detail, is unable to achieve a single step in advance of its starting-point, but depends rather upon a misuse of such ideas as have been acquired without its aid, according to the well-tried principles of scientific procedure. Far from being a model of valid reasoning in general, it is a mistaken account of what is taken dogmatically to be the

special method of deduction itself, and entirely misses or ignores the complementary function of induction in the establishment of new concepts and theories.

The ultimate philosophical presuppositions of mathematical logic, then, we may conveniently summarize under three complementary heads: those, namely, of ontology, of linear inference (or 'logic'), and of epistemology.

Ontologically, the theory of reasoning which we have been examining involves a doctrine of atomistic pluralism and abstract individualism; of quantitative 'wholes' composed of a sum of independent 'parts'. And the parts are existential somewhats, bare particulars, whose only band of association consists of 'external' relations, after the analogy of objects in space. Hence the sphere of reality to which such conceptions belong is that of mere Being, the most abstract and inadequate viewpoint which it is possible for thought to contemplate. In fact, the only attitude which may be adopted towards the objects of this realm, it seems, is that of mute pointing, of passive acceptance of the immediately given. No wonder thought becomes dumb, under such circumstances, and is forced to have recourse either to the neutral, indescribable entities of neo-realism or the absolutely homogeneous units (of the 'base-system') of Josiah Royce!

So much for the elementary material, of which, literally speaking, everything *ex hypothesi* is composed. Now mathematical logic—absolutely formal logic, operating by means of symbols—will have for its complementary task, in this regard, to discover and systematically present the various empty forms into which any sort of material may be arranged. Since the form and the content of knowledge are absolutely distinct, the only movement of thought is that of a linear progression, according to the law of abstract identity or tautology, while the result of such a process is a mechanical repetition or reshuffling of the meaningless counters or symbols, with which it began. Like Descartes and Locke, the mathematical logicians totally ignore the inception, growth and development of new conceptions, in which the vitality of scientific thought expresses itself. For this type or rationalism, "a thing, a mind, an idea, a principle is what it is unchangeably; so that either, as in the case of Descartes, the variety of real thought is

contained, perfect and entire, within its unity, and is to be set forth by pure subsumption, the lifting out of class from within class, or, as in the case of Locke, the unity of real thought is a mere aggregate of its varieties, the elements remaining unchangeable into whatever groups we may gather them."²³

Having separated thought and its object, form and content, we may study each in isolation from the other; but sooner or later we must also show, it appears, how after all to get these two moments of experience together again. This last problem, that of epistemology, so-called, is perhaps the most difficult of the three. Indeed, there are good grounds for considering insoluble this problem of determining how subjective process and external material happen to come together in the shape of genuine knowledge.

It is no part of our purpose to enter deeply into this problem here. Enough to observe that mathematical logic is always associated with some theory of 'belief' (Mr. Russell), or an arbitrary 'act of will' (Josiah Royce), by which to relate the demonstrative truths of logic and pure mathematics to the real world. But beliefs and acts of will are unavoidably subjective; there presumably may be other kinds of beings who believe and will differently from ourselves. Hence the very disease to be avoided, subjectivity, attaches itself to, and infects, these doctrines. In the attempt to avoid 'mind,' as tainted with subjectivity, Mr. Russell separates the 'real world' wholly from it, only to be forced, in the end, to give an account, even if incoherent, of the relationship between these two factors. And in the endeavor to distinguish the ideal from the merely empirical, Royce likewise is led to separate the ideal world of 'pure forms' from the world of physical existence. The end result, as we have seen, is essentially the same, in both cases, though expressed in slightly different language.

With such presuppositions, we may confidently assert, neither science nor philosophy, in the proper sense of those words, has any concern—except to demonstrate, in the interests of the pursuit of objective truth, their essentially alogical and inescapably subjective character.

²³ Latta, *Leibniz*, p. 128. Cf. also Cassirer, *Substance and Function*, Eng. transl., pp. 316 ff.; Russell, *Scientific Method in Philosophy*, pp. 181 ff., 202, and *passim*.

